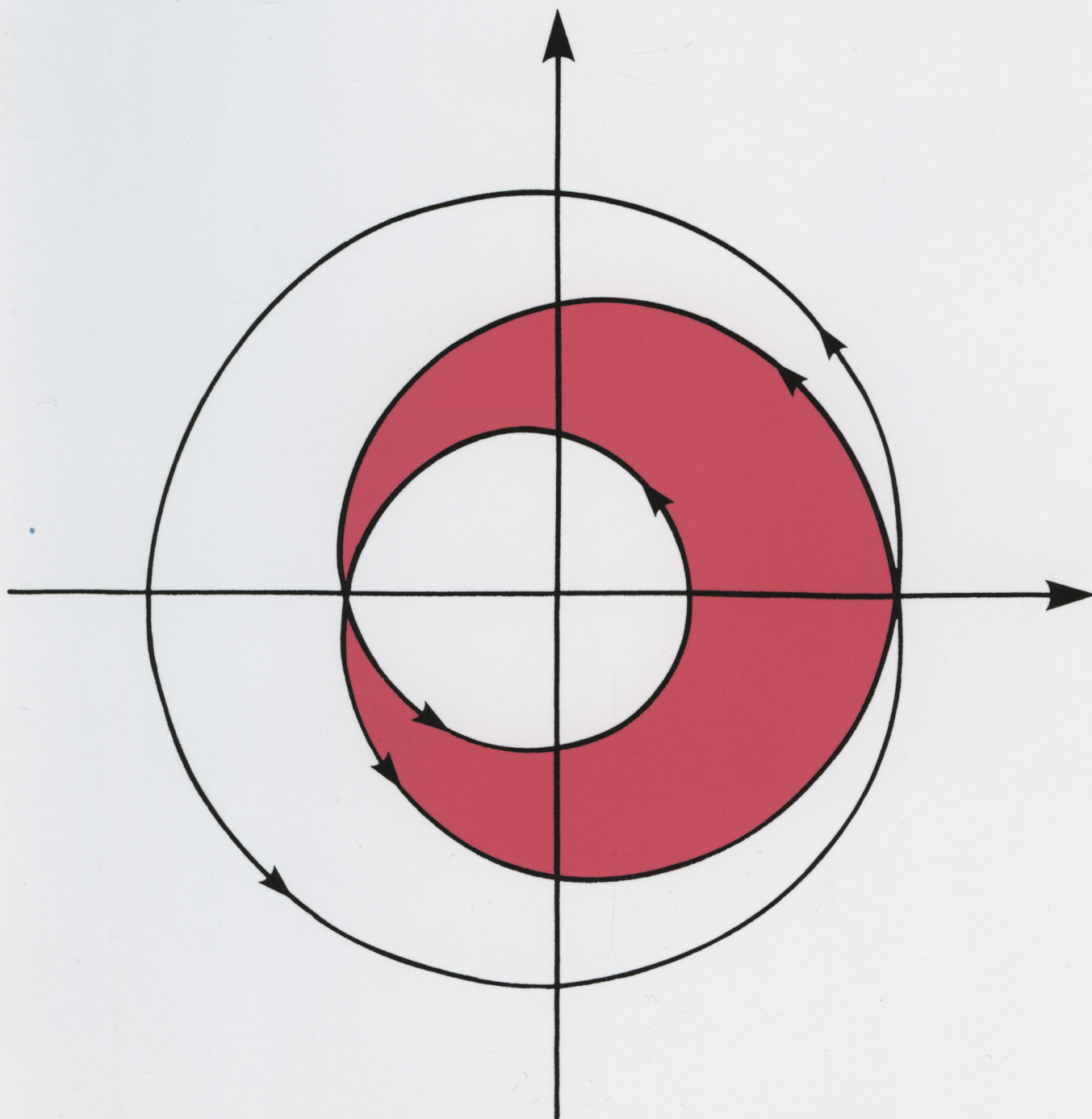


# COMPLEX ANALYSIS

## UNIT C1 RESIDUES





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## UNIT C1 RESIDUES

*Prepared by the Course Team*

The Open University, Walton Hall, Milton Keynes, MK7 6AA  
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COMPLEX ANALYSIS COMPLEX ANALYSIS COMPLEX ANALYSIS



Before working through this text, make sure that you have read the  
*Course Guide* for M337 Complex Analysis.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

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# INTRODUCTION

In Block B we obtained many theoretical results about analytic functions and we pointed out that these can often be 'turned round' to yield methods of evaluating integrals. For example, if a function  $f$  is analytic on a simply-connected region  $\mathcal{R}$ , if  $\Gamma$  is a simple-closed contour in  $\mathcal{R}$ , and if  $\alpha$  is any point inside  $\Gamma$ , then

$$\int_{\Gamma} f(z) dz = 0 \quad (\text{by Cauchy's Theorem}), \quad \text{Unit B2, Theorem 1.2}$$

$$\int_{\Gamma} \frac{f(z)}{z - \alpha} dz = 2\pi i f(\alpha) \quad (\text{by Cauchy's Integral Formula}), \quad \text{Unit B2, Theorem 2.1}$$

and, for  $n = 1, 2, \dots$ ,

$$\int_{\Gamma} \frac{f(z)}{(z - \alpha)^{n+1}} dz = 2\pi i \frac{f^{(n)}(\alpha)}{n!} \quad (\text{by Cauchy's } n\text{th Derivative Formula}). \quad \text{Unit B2, Theorem 3.2}$$

If, on the other hand,  $f$  is analytic only on the punctured disc  $D = \{z : 0 < |z - \alpha| < r\}$ , then

$$\int_C \frac{f(z)}{(z - \alpha)^{n+1}} dz = 2\pi i a_n,$$

where  $a_n$  is the coefficient of  $(z - \alpha)^n$  in the Laurent series about  $\alpha$  for  $f$ , and  $C$  is any circle in  $D$  with centre  $\alpha$ . In particular, when  $n = -1$ , we have

$$\int_C f(z) dz = 2\pi i a_{-1}; \quad (0.1) \quad \text{Unit B4, Equation (4.2)}$$

$a_{-1}$  is called the *residue of  $f$  at  $\alpha$* , and is denoted by  $\text{Res}(f, \alpha)$ .

In this unit we use residues to evaluate more general complex integrals, and hence solve a number of problems from real analysis. We start, in Section 1, by introducing a number of useful techniques for evaluating residues.

In Section 2 we state and prove a key result called the Residue Theorem, which can be thought of as a generalization of Equation (0.1) to a simple-closed contour surrounding a *finite number* of singularities.

We then apply the Residue Theorem to the evaluation of a wide range of integrals. In particular, we show how it can be used to evaluate certain *real trigonometric integrals* involving  $\cos t$  and  $\sin t$ , such as

$$\int_0^{2\pi} \frac{1}{5 + 4 \sin t} dt \quad \text{and} \quad \int_0^{2\pi} \frac{1}{25 \cos^2 t + 9 \sin^2 t} dt.$$

In Section 3, we show you how to use the Residue Theorem to evaluate certain *real improper integrals* of the forms

$$\int_{-\infty}^{\infty} f(t) dt, \quad \int_{-\infty}^{\infty} f(t) \cos kt dt \quad \text{and} \quad \int_{-\infty}^{\infty} f(t) \sin kt dt,$$

where  $f$  is a rational function.

Finally, in Section 4, you will see how residues can also be used to determine the sums of certain real infinite series. For example, we shall show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

All these applications of the Residue Theorem will, we hope, convince you of the truth of these words of Cauchy, taken from the introduction to his 1826 paper 'Sur un nouveau genre de calcul':

*Les résidus ... se présentent naturellement dans plusieurs branches de l'analyse ... Leur considération fournit des méthodes simples et d'un usage facile, qui s'appliquent à un grand nombre de questions diverses, et des formules nouvelles ...*

Residues arise ... naturally in several branches of analysis ... Their consideration provides simple and easy-to-use methods, which are applicable to a large number of diverse questions, and some new results ...



## Study guide

In Section 1 you should make sure that you are familiar with the various methods for calculating residues, as they will be needed throughout the rest of the unit.

In Section 2 you should make sure that you understand the statement and use of the Residue Theorem. If you are short of time, you may prefer to omit the proof of the Residue Theorem on a first reading, and proceed to Section 3.

Section 3 (the audio-tape section) forms the heart of this unit, and you will need to master it as the techniques introduced there will be needed in later units. It will probably take you longer to study than the other sections, so you should make sure that you allocate enough time for it.

Section 4 presents a rather different application of the Residue Theorem. Several of the results obtained depend on residues evaluated in Section 1.

## 1 CALCULATING RESIDUES

After working through this section, you should be able to:

- (a) calculate residues by determining the relevant term of the Laurent series;
- (b) calculate residues at simple poles and removable singularities, using the  $g/h$  Rule and the Cover-up Rule;
- (c) calculate residues at higher-order poles.

### 1.1 Using the Laurent series

Since the residue of an analytic function  $f$  at a singularity  $\alpha$  of  $f$  is the coefficient  $a_{-1}$  of  $(z - \alpha)^{-1}$  in the Laurent series for  $f$ :

$$f(z) = \cdots + \frac{a_{-2}}{(z - \alpha)^2} + \frac{a_{-1}}{z - \alpha} + a_0 + a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots,$$

we can determine  $\text{Res}(f, \alpha)$  simply by inspecting the series. The following example will show you some of the techniques involved.

#### Example 1.1

Find the residue of each of the following functions  $f$  at the point 0.

(a)  $f(z) = \frac{1 + z^2}{z}$       (b)  $f(z) = \frac{\sin z}{z^4}$

#### Solution

(a) The Laurent series for  $f$  is clearly

$$\frac{1 + z^2}{z} = \cdots + \frac{0}{z^3} + \frac{0}{z^2} + \frac{1}{z} + 0 + z + 0z^2 + 0z^3 + \cdots,$$

and so

$$\text{Res}(f, 0) = \text{the coefficient of } z^{-1} = 1.$$

(b) The Taylor series about 0 for the sine function is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots;$$

dividing by  $z^4$ , we obtain the Laurent series about 0 for  $f$ :

$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \cdots,$$

and so  $\text{Res}(f, 0) = \text{the coefficient of } z^{-1} = -\frac{1}{6}$ . ■



### Problem 1.1

Find the residue of each of the following functions  $f$  at 0.

(a)  $f(z) = \frac{1}{z^2} - 3$       (b)  $f(z) = \frac{1}{z-1}$

(c)  $f(z) = \frac{\cos z}{z^3}$       (d)  $f(z) = z^2 \sin\left(\frac{1}{z}\right)$

It sometimes helps to rearrange the function into a more useful form when you are finding residues.

### Example 1.2

Find the residues of the function  $f(z) = \frac{1}{z(z-1)^2}$  at each of the following points  $\alpha$ .

(a)  $\alpha = 0$       (b)  $\alpha = 1$

### Solution

(a) We need to find the Laurent series about 0 for  $f$ . We have, by the Binomial Theorem,

Unit B3, Frame 1

$$\begin{aligned}\frac{1}{z(z-1)^2} &= \frac{1}{z}(1-z)^{-2} \\ &= \frac{1}{z}(1+2z+3z^2+\cdots), \quad \text{for } 0 < |z| < 1, \\ &= \frac{1}{z} + 2 + 3z + \cdots,\end{aligned}$$

and so

$$\text{Res}(f, 0) = \text{the coefficient of } z^{-1} = 1.$$

(b) We need to expand the given function in powers of  $z-1$ . To do this, we may substitute  $z-1 = h$ , so that  $z = 1+h$ . We obtain, for  $z \neq 0, 1$ ,

$$\begin{aligned}\frac{1}{z(z-1)^2} &= \frac{1}{(1+h)h^2} \\ &= \frac{1}{h^2}(1+h)^{-1} \\ &= \frac{1}{h^2}(1-h+h^2-h^3+\cdots), \quad \text{for } 0 < |h| < 1, \\ &= \frac{1}{h^2} - \frac{1}{h} + 1 - h + \cdots \\ &= \frac{1}{(z-1)^2} - \frac{1}{(z-1)} + 1 - (z-1) + \cdots, \quad \text{for } 0 < |z-1| < 1.\end{aligned}$$

Thus

$$\text{Res}(f, 1) = \text{the coefficient of } (z-1)^{-1} = -1. \quad \blacksquare$$

Notice that, since the residue of  $f$  at the point  $\alpha$  is the coefficient of  $(z-\alpha)^{-1}$  in the Laurent series about  $\alpha$  for  $f$ , it follows, on writing  $z = \alpha + h$ , that this residue is simply the coefficient of  $h^{-1}$  in the resulting Laurent series about 0. You can use this observation in the following problem.

### Problem 1.2

Find the residue of each of the following functions  $f$  at the given point  $\alpha$ .

(a)  $f(z) = \frac{1}{z^2+1}$ ,  $\alpha = i$       (b)  $f(z) = \frac{ze^{iz}}{(z-\pi)^2}$ ,  $\alpha = \pi$



If we wished to calculate the residue at the point  $\frac{1}{3}$  of the function

$$f(z) = \frac{1}{z^2(1-z)(1-2z)(1-3z)},$$

then we would be faced with the complicated task of calculating the coefficient of  $(z - \frac{1}{3})^{-1}$  in the Laurent series about  $\frac{1}{3}$  for  $f$ . Fortunately, there are a number of useful rules for determining residues at simple poles and removable singularities. We turn our attention to these rules in the next subsection.

## 1.2 Methods for simple poles and removable singularities

Suppose that the analytic function  $f$  has a simple pole at the point  $\alpha$ . Then  $f$  can be represented by a Laurent series of the form

$$f(z) = \frac{a_{-1}}{z - \alpha} + a_0 + a_1(z - \alpha) + \dots, \quad \text{where } a_{-1} \neq 0,$$

on a punctured open disc centred at  $\alpha$ .

If we multiply this equation by  $z - \alpha$ , then we obtain

$$(z - \alpha)f(z) = a_{-1} + a_0(z - \alpha) + a_1(z - \alpha)^2 + \dots$$

The power series on the right defines a function which is analytic, and therefore continuous, at the point  $\alpha$ . We can therefore let  $z$  tend to  $\alpha$ , giving

$$\lim_{z \rightarrow \alpha} (z - \alpha)f(z) = a_{-1} = \text{Res}(f, \alpha). \quad (1.1)$$

If

$$g(z) = a_{-1} + a_0(z - \alpha) + \dots,$$

then  $g(\alpha) = a_{-1}$ .

Note that a similar conclusion holds when the function  $f$  has a removable singularity at the point  $\alpha$ . The only difference is that  $a_{-1} = 0$ , and we deduce that

$$\lim_{z \rightarrow \alpha} (z - \alpha)f(z) = \text{Res}(f, \alpha) = 0.$$

These results for simple poles and removable singularities suggest the following result.

**Theorem 1.1** Suppose that an analytic function  $f$  has a singularity at the point  $\alpha$ . Then

$$\text{Res}(f, \alpha) = \lim_{z \rightarrow \alpha} (z - \alpha)f(z),$$

provided that this limit exists.

**Proof** Let  $\beta = \lim_{z \rightarrow \alpha} (z - \alpha)f(z)$ .

If  $\beta = 0$ , then  $f$  has a removable singularity at  $\alpha$  (by Theorem 3.1 of Unit B4), and so  $\text{Res}(f, \alpha) = 0$ .

On the other hand, if  $\beta \neq 0$ , then  $f$  has a simple pole at  $\alpha$  (by Theorem 3.2 of Unit B4), and so  $\text{Res}(f, \alpha) = \beta$ , by Equation (1.1). ■

We use this result in the following example.

### Example 1.3

Find the residue of each of the following functions  $f$  at the given point  $\alpha$ .

$$(a) f(z) = \frac{1}{z(z-1)^2}, \alpha = 0 \quad (b) f(z) = \frac{1}{z^2+1}, \alpha = i$$

#### Solution

(a) Since

$$\lim_{z \rightarrow 0} z \cdot \frac{1}{z(z-1)^2} = \lim_{z \rightarrow 0} \frac{1}{(z-1)^2} = 1,$$

we deduce that  $\text{Res}(f, 0) = 1$ .

(b) Since

$$\lim_{z \rightarrow i} (z-i) \cdot \frac{1}{z^2+1} = \lim_{z \rightarrow i} \frac{1}{z+i} = \frac{1}{2i},$$

we deduce that  $\text{Res}(f, i) = -\frac{1}{2}i$ . ■

You should compare these solutions with those of Example 1.2(a) and Problem 1.2(a).

### Problem 1.3

Find the residue of each of the following functions  $f$  at the given point  $\alpha$ .

$$(a) f(z) = \frac{1}{z^2+4}, \alpha = 2i$$

$$(b) f(z) = \frac{1}{z^2(1-z)(1-2z)(1-3z)}, \alpha = \frac{1}{3}$$

The function in part (b) was mentioned at the end of Subsection 1.1.

A useful corollary to Theorem 1.1 is the following result, which we call the  $g/h$  Rule.

#### Corollary 1 $g/h$ Rule

Let  $f(z) = g(z)/h(z)$ , where the functions  $g$  and  $h$  are analytic at the point  $\alpha$ ,  $h(\alpha) = 0$ , and  $h'(\alpha) \neq 0$ . Then

$$\text{Res}(f, \alpha) = g(\alpha)/h'(\alpha).$$

**Proof** We have

$$\begin{aligned} \lim_{z \rightarrow \alpha} (z-\alpha) \frac{g(z)}{h(z)} &= \lim_{z \rightarrow \alpha} g(z) \cdot \frac{z-\alpha}{h(z)-h(\alpha)} \quad (\text{since } h(\alpha) = 0) \\ &= \lim_{z \rightarrow \alpha} g(z) / \lim_{z \rightarrow \alpha} \frac{h(z)-h(\alpha)}{z-\alpha} \\ &= g(\alpha)/h'(\alpha); \end{aligned}$$

Note the use here of the Quotient Rule for limits (Unit A3, Theorem 3.2); this is valid because  $h'(\alpha) \neq 0$ .

thus the limit exists and the result follows from Theorem 1.1. ■

### Example 1.4

Find the residues of the function  $f(z) = \frac{z^2}{z^4-1}$  at the point  $i$ .

#### Solution

Let

$$g(z) = z^2 \quad \text{and} \quad h(z) = z^4 - 1.$$

Then the functions  $g$  and  $h$  are analytic at  $i$ . Also,  $h(i) = i^4 - 1 = 0$ , and  $h'(i) = 4i^3$ , which is non-zero. Thus the  $g/h$  Rule applies, and we have

$$\begin{aligned} \text{Res}(f, i) &= g(i)/h'(i) \\ &= i^2/(4i^3) = -\frac{1}{4}i. \quad \blacksquare \end{aligned}$$



### Problem 1.4

Find the residue of each of the following functions  $f$  at the given points  $\alpha$ .

(a)  $f(z) = \frac{1}{2z^2 + 5iz - 2}$ ,  $\alpha = -\frac{1}{2}i$

(b)  $f(z) = \frac{z+9}{(z^2+1)(z^2+9)}$ ,  $\alpha = 3i$

(c)  $f(z) = \frac{z^3}{z^4+1}$ ,  $\alpha = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}$  and  $e^{7i\pi/4}$

The  $g/h$  Rule is particularly useful in examples where the function  $h$  is a trigonometric or exponential function. The following examples, whose solutions will be needed in Section 4, are good instances of this.

### Example 1.5

Find the residues of each of the following functions  $f$  at the point  $n$ , where  $n$  is any integer.

(a)  $f(z) = \pi \cot \pi z$       (b)  $f(z) = \pi \operatorname{cosec} \pi z$

#### Solution

(a) Let

$$g(z) = \pi \cos \pi z \quad \text{and} \quad h(z) = \sin \pi z.$$

Then the functions  $g$  and  $h$  are analytic at  $n$ . Also,

$$h(n) = \sin \pi n = 0,$$

and

$$h'(n) = \pi \cos \pi n, \text{ which is non-zero.}$$

Thus the  $g/h$  Rule applies, and we obtain

$$\begin{aligned} \operatorname{Res}(f, n) &= g(n)/h'(n) \\ &= \frac{\pi \cos \pi n}{\pi \cos \pi n} = 1. \end{aligned}$$

(b) Let

$$g(z) = \pi \quad \text{and} \quad h(z) = \sin \pi z.$$

Then the functions  $g$  and  $h$  are analytic at  $n$ . Also,

$$h(n) = \sin \pi n = 0,$$

and

$$h'(n) = \pi \cos \pi n, \text{ which is non-zero.}$$

Thus the  $g/h$  Rule applies, and we obtain

$$\begin{aligned} \operatorname{Res}(f, n) &= g(n)/h'(n) \\ &= \frac{\pi}{\pi \cos \pi n} \\ &= \frac{1}{(-1)^n} = (-1)^n. \quad \blacksquare \end{aligned}$$

### Problem 1.5

Find the residue of each of the following functions  $f$  at the given points  $\alpha$ .

(a)  $f(z) = \frac{\pi \operatorname{cosec} \pi z}{4z^2 - 1}$ ,  $\alpha = \frac{1}{2}$  and  $-\frac{1}{2}$

(b)  $f(z) = \frac{\pi \cot \pi z}{4z^2 - 1}$ ,  $\alpha = \frac{1}{2}$  and  $-\frac{1}{2}$

(c)  $f(z) = \frac{\pi \operatorname{cosec} \pi z}{4z^2 + 1}$ ,  $\alpha = \frac{1}{2}i$  and  $-\frac{1}{2}i$

(d)  $f(z) = \frac{\pi \cot \pi z}{4z^2 + 1}$ ,  $\alpha = \frac{1}{2}i$  and  $-\frac{1}{2}i$

These residues will be needed in Section 4 and the associated exercises.

Next we state another corollary of Theorem 1.1 which is sometimes convenient.

#### Corollary 2 Cover-up Rule

Let  $f(z) = \frac{p(z)}{(z - \alpha)q(z)}$ , where the functions  $p$  and  $q$  are analytic at the point  $\alpha$ , and  $q(\alpha) \neq 0$ . Then

$$\operatorname{Res}(f, \alpha) = p(\alpha)/q(\alpha).$$

The reason for this name is explained below.

**Proof** Since  $q(\alpha) \neq 0$ , we have

$$\begin{aligned} \lim_{z \rightarrow \alpha} (z - \alpha)f(z) &= \lim_{z \rightarrow \alpha} \frac{p(z)}{q(z)} \\ &= p(\alpha)/q(\alpha); \end{aligned}$$

so the result follows from Theorem 1.1. ■

The reason for the name 'Cover-up Rule' is that we can find the residue of a function at the simple pole  $\alpha$  by covering up the factor  $z - \alpha$  in the denominator and evaluating what remains at the point  $\alpha$ . Like Theorem 1.1 and Corollary 1, this method works *only for simple poles and removable singularities*.

Take care to cover up only a factor of the form  $z - \alpha$ .

### Example 1.6

Find the residues of the function

$$f(z) = \frac{3z}{(z - i)(2z - i)}$$

at the points  $i$  and  $\frac{1}{2}i$ .

#### Solution

The function  $f$  has simple poles at the points  $i$  and  $\frac{1}{2}i$ . If we cover up the term  $(z - i)$  and evaluate what remains at  $i$ , then we obtain

$$\begin{aligned} \operatorname{Res}(f, i) &= \frac{3i}{(z - i)(2i - i)} \\ &= \frac{3i}{i} = 3. \end{aligned}$$

We cannot similarly cover up the term  $(2z - i)$ , as this is not of the required form  $(z - \alpha)$ . We therefore write

$$f(z) = \frac{3z}{2(z - i)(z - \frac{1}{2}i)}.$$



If we now cover up the term  $(z - \frac{1}{2}i)$  and evaluate what remains at  $\frac{1}{2}i$ , then we obtain

$$\begin{aligned}\operatorname{Res}(f, \tfrac{1}{2}i) &= \frac{3 \cdot (\frac{1}{2}i)}{2(\frac{1}{2}i - i)(z - \frac{1}{2}i)} \\ &= \frac{\frac{3}{2}i}{2(-\frac{1}{2}i)} = -\frac{3}{2}. \quad \blacksquare\end{aligned}$$

### Problem 1.6

Use the Cover-up Rule to find the residue of each of the following functions  $f$  at the given point  $\alpha$ .

(a)  $f(z) = \frac{z+2}{z^3(z+4)}, \alpha = -4$       (b)  $f(z) = \frac{\cos z}{ze^z}, \alpha = 0$

(c)  $f(z) = \frac{1}{z^2(1-z)(1-2z)(1-3z)}, \alpha = \frac{1}{3}$       (d)  $f(z) = \frac{z^2 \sin z}{(e^z - 1)^3}, \alpha = 0$

## 1.3 Methods for higher-order poles

Generally speaking, the calculation of residues at poles of order greater than one is an unpleasant chore, as the useful rules of Subsection 1.2 (the  $g/h$  Rule and the Cover-up Rule) are no longer valid. There are essentially two methods for dealing with higher-order poles — using the Laurent series directly, and using a ‘higher-order analogue’ of Theorem 1.1. Which method is more appropriate will depend on the nature of the function in question.

### Using the Laurent series

You have already seen how Laurent series can be used to find residues at poles of order greater than one. For example, in Examples 1.1(b) and 1.2(b), we used Laurent series to calculate the residues of

$$f(z) = \frac{\sin z}{z^4} \text{ at } 0, \quad \text{and} \quad f(z) = \frac{1}{z(z-1)^2} \text{ at } 1.$$

Here are two more such residues for you to calculate.

### Problem 1.7

Use the appropriate Laurent series to find the residue of each of the following functions  $f$  at the given point  $\alpha$ .

(a)  $f(z) = \frac{z+2}{z^3(z+4)}, \alpha = 0$       (b)  $f(z) = \frac{1+e^{2z}}{(z-1)^4}, \alpha = 1$

## The 'higher-order formula'

If  $f$  is a function with a pole of order two at the point  $\alpha$ , then its Laurent series about  $\alpha$  can be written in the form

$$f(z) = \frac{a_{-2}}{(z-\alpha)^2} + \frac{a_{-1}}{z-\alpha} + a_0 + a_1(z-\alpha) + \cdots, \quad \text{where } a_{-2} \neq 0.$$

We wish to isolate the term  $a_{-1}$ . It is tempting just to multiply through by  $z-\alpha$  and then let  $z$  tend to  $\alpha$ , but this would cause problems with the first term. Instead, we multiply through by  $(z-\alpha)^2$ , giving

$$(z-\alpha)^2 f(z) = a_{-2} + a_{-1}(z-\alpha) + a_0(z-\alpha)^2 + a_1(z-\alpha)^3 + \cdots.$$

If we now differentiate this equation, then we obtain

$$\frac{d}{dz}((z-\alpha)^2 f(z)) = a_{-1} + 2a_0(z-\alpha) + 3a_1(z-\alpha)^2 + \cdots.$$

Taking the limit as  $z$  tends to  $\alpha$ , we deduce that the residue of  $f$  at  $\alpha$  is given by

$$a_{-1} = \lim_{z \rightarrow \alpha} \left( \frac{d}{dz}((z-\alpha)^2 f(z)) \right).$$

Using a similar method, we can prove the corresponding formula for a pole of general order  $k$ .

**Theorem 1.2** If a function  $f$  has a pole of order  $k$  at the point  $\alpha$ , then the residue of  $f$  at  $\alpha$  is given by

$$\text{Res}(f, \alpha) = \frac{1}{(k-1)!} \lim_{z \rightarrow \alpha} \left( \frac{d^{k-1}}{dz^{k-1}} ((z-\alpha)^k f(z)) \right).$$

You are asked to prove Theorem 1.2 in Problem 1.9 below.

### Example 1.7

Find the residue of the function  $f(z) = \frac{1+e^{2z}}{(z-1)^4}$  at the point 1.

#### Solution

Since the function  $f$  has a pole of order four at the point 1, we apply Theorem 1.2 with  $k=4$ . We obtain

$$\begin{aligned} \text{Res}(f, 1) &= \frac{1}{3!} \lim_{z \rightarrow 1} \left( \frac{d^3}{dz^3} (1+e^{2z}) \right) \\ &= \frac{1}{6} \lim_{z \rightarrow 1} 8e^{2z} = \frac{8}{6} e^2 = \frac{4}{3} e^2. \quad \blacksquare \end{aligned}$$

Note that this answer agrees with that of Problem 1.7(b).

### Problem 1.8

Use Theorem 1.2 to find the residue of each of the following functions  $f$  at the given point  $\alpha$ .

$$(a) f(z) = \frac{ze^{iz}}{(z-\pi)^2}, \alpha = \pi \quad (b) f(z) = \frac{z+2}{z^3(z+4)}, \alpha = 0$$

### Problem 1.9

Prove Theorem 1.2.



## 2 THE RESIDUE THEOREM

After working through this section, you should be able to:

- state Cauchy's Residue Theorem, and use it to evaluate a given contour integral;
- evaluate real trigonometric integrals of the form  $\int_0^{2\pi} \Phi(\cos t, \sin t) dt$ , where  $\Phi$  is a function of two variables.

### 2.1 Cauchy's Residue Theorem

At the end of *Unit B4* you saw that if  $f$  is a function which is analytic on the punctured disc  $D = \{z : 0 < |z - \alpha| < r\}$ , then

$$\int_C f(z) dz = 2\pi i \operatorname{Res}(f, \alpha),$$

where  $C$  is any circle in  $D$  with centre  $\alpha$ . It follows that we can evaluate such an integral by calculating the residue and multiplying the result by  $2\pi i$ .

We now extend this result to the case where  $f$  has several singularities at points inside a simple-closed contour  $\Gamma$ . In this case, each singularity  $\alpha$  gives a contribution of  $2\pi i \operatorname{Res}(f, \alpha)$  to the value of the integral; we then evaluate the integral of  $f$  along  $\Gamma$  by adding up these contributions, giving

$$2\pi i \times (\text{the sum of the residues at the singularities inside } \Gamma).$$

The formal statement of this result is as follows; its proof is given at the end of the section.

#### Theorem 2.1 Cauchy's Residue Theorem

Let  $\mathcal{R}$  be a simply-connected region, and let  $f$  be a function which is analytic on  $\mathcal{R}$  except for a finite number of singularities. Let  $\Gamma$  be any simple-closed contour in  $\mathcal{R}$ , not passing through any of these singularities. Then

$$\int_{\Gamma} f(z) dz = 2\pi i S,$$

where  $S$  is the sum of the residues of  $f$  at those singularities that lie inside  $\Gamma$ .

Note that, as usual, it is convenient to refer to  $\alpha$  both as a singularity and as a point.

Notice that if no singularities of  $f$  lie at points inside  $\Gamma$ , then the Residue Theorem reduces to a special case of Cauchy's Theorem, and the value of the integral is 0.

#### Remarks

- We usually refer to Theorem 2.1 simply as 'the Residue Theorem'.
- When the Residue Theorem is used to evaluate an integral, it is good practice to draw the contour  $\Gamma$  and plot the singularities of the function  $f$ , for then it is easy to identify those singularities which lie *inside*  $\Gamma$ . If any of the singularities lie on  $\Gamma$ , then the integral is not defined; for example,

$$\int_{\Gamma} \frac{1}{z(z-1)^2} dz, \quad \text{where } \Gamma = \{z : |z| = 1\},$$

is not defined (see Figure 2.1).

The Residue Theorem can be used to evaluate integrals to which we applied the partial fractions strategy of *Unit B2*, Frame 8 (see, for example, Example 2.1). However, it provides a much more general method (see, for example, Problem 2.4, to which the *Unit B2* strategy cannot be applied). The following examples illustrate the basic method.

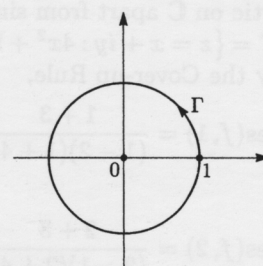


Figure 2.1

### Example 2.1

Evaluate  $\int_{\Gamma} \frac{1}{z(z-1)^2} dz$ , where  $\Gamma$  is

- (a) the circle  $\{z : |z-1| = \frac{1}{2}\}$ ;
- (b) any rectangular contour containing both 0 and 1 in its inside.

### Solution

(a) The function

$$f(z) = \frac{1}{z(z-1)^2}$$

is analytic on  $\mathbb{C}$  apart from singularities at 0 and 1; 0 lies outside  $\Gamma = \{z : |z-1| = \frac{1}{2}\}$  and 1 lies inside  $\Gamma$  (see Figure 2.2). By Example 1.2(b),  $\text{Res}(f, 1) = -1$ .

Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ ,

$$\int_{\Gamma} f(z) dz = 2\pi i \text{Res}(f, 1) = -2\pi i.$$

(b) The function

$$f(z) = \frac{1}{z(z-1)^2}$$

is analytic on  $\mathbb{C}$  apart from singularities at 0 and 1, both of which lie inside  $\Gamma$  (see Figure 2.3). As in part (a),  $\text{Res}(f, 1) = -1$ , and by Example 1.2(a) or the Cover-up Rule,  $\text{Res}(f, 0) = 1$ .

Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ ,

$$\begin{aligned} \int_{\Gamma} f(z) dz &= 2\pi i (\text{Res}(f, 1) + \text{Res}(f, 0)) \\ &= 2\pi i (-1 + 1) = 0. \quad \blacksquare \end{aligned}$$

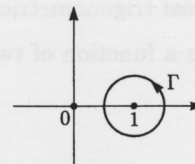


Figure 2.2

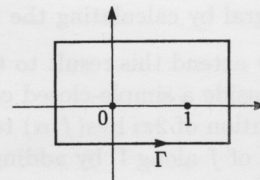


Figure 2.3

### Example 2.2

Evaluate the integral

$$\int_{\Gamma} \frac{z+3}{(z-1)(z-2)(z+4)} dz,$$

where  $\Gamma$  is the ellipse  $\{z = x + iy : 4x^2 + 9y^2 = 36\}$ .

### Solution

The function

$$f(z) = \frac{z+3}{(z-1)(z-2)(z+4)}$$

is analytic on  $\mathbb{C}$  apart from singularities at 1, 2 and  $-4$ ; of these 1 and 2 lie inside  $\Gamma = \{z = x + iy : 4x^2 + 9y^2 = 36\}$  and  $-4$  lies outside  $\Gamma$  (see Figure 2.4). Now, by the Cover-up Rule,

$$\text{Res}(f, 1) = \frac{1+3}{(1-2)(1+4)} = -\frac{4}{5},$$

and

$$\text{Res}(f, 2) = \frac{2+3}{(2-1)(2+4)} = \frac{5}{6}.$$

Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ ,

$$\begin{aligned} \int_{\Gamma} f(z) dz &= 2\pi i (\text{Res}(f, 1) + \text{Res}(f, 2)) \\ &= 2\pi i \left( -\frac{4}{5} + \frac{5}{6} \right) = \frac{\pi i}{15}. \quad \blacksquare \end{aligned}$$

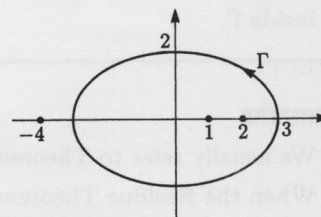


Figure 2.4

Although  $f$  has a pole at  $-4$ , we do not need to calculate its residue, as  $-4$  lies outside  $\Gamma$ .



**Problem 2.1**

Evaluate the integral  $\int_{\Gamma} \frac{\sin z}{z^2 - 1} dz$ , where  $\Gamma$  is

- (a) the circle  $\{z : |z| = 3\}$ ;  
 (b) the rectangular contour with vertices at  $-2i, 2i, -2 + 2i, -2 - 2i$ .

**Problem 2.2**

Let  $\Gamma$  be the circle  $\{z : |z - i| = 2\}$ . Evaluate the integral

$$I = \int_{\Gamma} \frac{z + 2}{4z^2 + k^2} dz,$$

- (a) when  $k = 1$ ; (b) when  $k = 3$ ; (c) when  $k = 7$ .

**Problem 2.3**

Using the result of Problem 1.4(c), evaluate the integral

$$\int_{\Gamma} \frac{z^3}{z^4 + 1} dz,$$

where  $\Gamma$  is the semicircular contour shown in Figure 2.5.

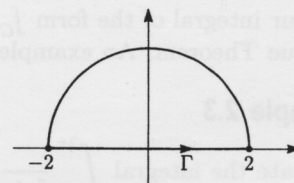


Figure 2.5

**Problem 2.4**

Evaluate the integral

$$\int_{\Gamma} \frac{1 + z}{\sin z} dz,$$

where  $\Gamma$  is the square contour with vertices  $4 + 4i, -4 + 4i, -4 - 4i, 4 - 4i$ .

Think carefully about the choice of  $\mathcal{R}$  in this problem.

## 2.2 Real trigonometric integrals

The Residue Theorem proves to be useful in some quite unexpected places. One of these is in the evaluation of real integrals of the form

$$\int_0^{2\pi} \Phi(\cos t, \sin t) dt,$$

where  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ . For example, if  $\Phi(x, y) = \frac{1}{4x^2 + y^2}$ , then

$$\int_0^{2\pi} \Phi(\cos t, \sin t) dt = \int_0^{2\pi} \frac{1}{4 \cos^2 t + \sin^2 t} dt.$$

To see how these are related to contour integrals, we consider first the contour integral

$$\int_C \frac{1}{2z^2 + 5iz - 2} dz,$$

where  $C$  is the unit circle  $\{z : |z| = 1\}$ .

If we tried to evaluate this contour integral by using the parametrization

$$\gamma(t) = e^{it} \quad (t \in [0, 2\pi]), \quad (2.1)$$

then we would get

$$\begin{aligned} \int_C \frac{1}{2z^2 + 5iz - 2} dz &= \int_0^{2\pi} \frac{ie^{it}}{2e^{2it} + 5ie^{it} - 2} dt \\ &= \int_0^{2\pi} \frac{i}{2e^{it} + 5i - 2e^{-it}} dt \\ &= \int_0^{2\pi} \frac{1}{5 + 2(e^{it} - e^{-it})/i} dt \\ &= \int_0^{2\pi} \frac{1}{5 + 4 \sin t} dt. \end{aligned}$$

$$\gamma'(t) = ie^{it}$$

Recall that  
 $\sin t = \frac{1}{2i}(e^{it} - e^{-it}).$

We now turn the whole argument around, transforming the given real trigonometric integral into a contour integral around the unit circle  $C$ . We can then evaluate this contour integral by using the Residue Theorem. Since Equation (2.1) amounts to substituting  $z = e^{it}$ , we need to replace

$$\cos t \text{ by } \frac{1}{2}(z + z^{-1}), \text{ since } z + z^{-1} = e^{it} + e^{-it} = 2 \cos t;$$

$$\sin t \text{ by } \frac{1}{2i}(z - z^{-1}), \text{ since } z - z^{-1} = e^{it} - e^{-it} = 2i \sin t;$$

and

$$dt \text{ by } \frac{1}{iz} dz, \text{ since } \frac{dz}{dt} = ie^{it} = iz.$$

By this means we replace the trigonometric integral  $\int_0^{2\pi} \Phi(\cos t, \sin t) dt$  by a contour integral of the form  $\int_C f(z) dz$ , which we can evaluate using the Residue Theorem. An example will make the method clear.

### Example 2.3

Evaluate the integral  $\int_0^{2\pi} \frac{1}{5 + 4 \sin t} dt$ .

### Solution

If  $C$  is the unit circle, then (by the above discussion)

$$\begin{aligned} \int_0^{2\pi} \frac{1}{5 + 4 \sin t} dt &= \int_C \frac{1}{5 + 4(z - z^{-1})/2i} \cdot \frac{1}{iz} dz \\ &= \int_C \frac{1}{5iz + 2(z^2 - 1)} dz \\ &= \int_C \frac{1}{2z^2 + 5iz - 2} dz, \end{aligned}$$

as expected.

Now,  $2z^2 + 5iz - 2 = (2z + i)(z + 2i)$ , and so the singularities of the function  $f(z) = 1/(2z^2 + 5iz - 2)$  are simple poles at  $-\frac{1}{2}i$  and  $-2i$ . Of these poles, that at  $-\frac{1}{2}i$  lies inside the unit circle  $C$ , and that at  $-2i$  lies outside  $C$ .

By the  $g/h$  Rule, with  $g(z) = 1$ ,  $h(z) = 2z^2 + 5iz - 2$ , and  $h'(z) = 4z + 5i$ ,

$$\operatorname{Res}(f, -\frac{1}{2}i) = \frac{1}{3i}.$$

This residue was calculated in Problem 1.4(a).

It follows from the Residue Theorem with  $\mathcal{R} = \mathbb{C}$  that the value of the required integral is

$$2\pi i \left( \frac{1}{3i} \right) = \frac{2}{3}\pi.$$

Note, as a check, that the answer is a real number. ■

We can summarize the above method in the form of a strategy.

### Strategy for evaluating $\int_0^{2\pi} \Phi(\cos t, \sin t) dt$

(a) Replace

$$\cos t \text{ by } \frac{1}{2}(z + z^{-1}), \sin t \text{ by } \frac{1}{2i}(z - z^{-1}) \text{ and } dt \text{ by } \frac{1}{iz} dz,$$

to obtain a contour integral of the form  $\int_C f(z) dz$  around the unit circle  $C = \{z : |z| = 1\}$ .

(b) Locate the singularities of the function  $f$  lying inside  $C$ , and calculate the residues of  $f$  at these points.

(c) Evaluate the given integral by calculating

$$2\pi i \times (\text{the sum of the residues found in step (b)}).$$



### Problem 2.5

Show that

$$\int_0^{2\pi} \frac{1}{25 \cos^2 t + 9 \sin^2 t} dt = \frac{1}{4i} \int_C \frac{z}{(z^2 + 4)(z^2 + \frac{1}{4})} dz,$$

where  $C$  is the unit circle  $\{z : |z| = 1\}$ .

Hence evaluate the given trigonometric integral.

### Problem 2.6

Let  $n$  be a positive integer.

- (a) Show that the residue of the function  $f(z) = (z^2 + 1)^n / z^{n+1}$  at the point 0 is

$$\begin{pmatrix} n \\ \frac{1}{2}n \end{pmatrix} \text{ if } n \text{ is even, and } 0 \text{ if } n \text{ is odd.}$$

- (b) Use this result to evaluate the integral

$$\int_0^{2\pi} \cos^n t dt.$$

Recall that

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}.$$

## 2.3 Proof of the Residue Theorem

We now prove the Residue Theorem.

### Theorem 2.1 Cauchy's Residue Theorem

Let  $\mathcal{R}$  be a simply-connected region, and let  $f$  be a function which is analytic on  $\mathcal{R}$  except for a finite number of singularities. Let  $\Gamma$  be any simple-closed contour in  $\mathcal{R}$ , not passing through any of these singularities. Then

$$\int_{\Gamma} f(z) dz = 2\pi i S,$$

where  $S$  is the sum of the residues of  $f$  at those singularities that lie inside  $\Gamma$ .

**Proof** Let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be the singularities of  $f$  lying *inside*  $\Gamma$ . For each singularity  $\alpha_k$ , choose a circle  $C_k$  with centre  $\alpha_k$  lying inside  $\Gamma$ , in such a way that the circles  $C_k$ ,  $k = 1, 2, \dots, n$ , do not meet (as illustrated in Figure 2.6 for the case  $n = 3$ ). Since  $f$  is analytic on a punctured open disc centred at  $\alpha_k$  containing  $C_k$ , we have, by Unit B4, Equation (4.2),

$$\int_{C_k} f(z) dz = 2\pi i \operatorname{Res}(f, \alpha_k), \quad \text{for } k = 1, 2, \dots, n. \quad (2.2)$$

Thus we wish to show that

$$\int_{\Gamma} f(z) dz = \sum_{k=1}^n \left( \int_{C_k} f(z) dz \right).$$

To do this, we express  $\int_{\Gamma} f(z) dz$  in terms of integrals around  $C_1, C_2, \dots, C_n$  and around other simple-closed contours, where the integral of  $f$  around each of these 'other' contours is zero (by Cauchy's Theorem).

This proof may be omitted on a first reading.

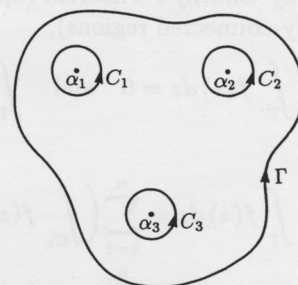


Figure 2.6 For simplicity, these circles have equal radii.

For each  $k = 1, 2, \dots, n$ , join  $C_k$  to  $\Gamma$  by two simple contours  $L_k$  and  $L'_k$  (for example, line segments) in such a way that no two of the contours  $L_1, L_2, \dots, L_n, L'_1, L'_2, \dots, L'_n$  meet (as indicated in Figure 2.7 for the case  $n = 3$ ).

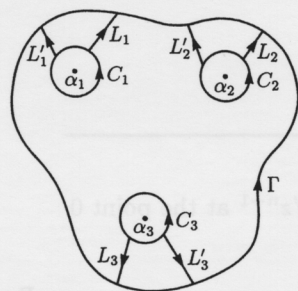


Figure 2.7

Also, let  $\Gamma_k, k = 1, 2, \dots, n$ , and  $\Gamma'$  be the simple-closed contours indicated in Figure 2.8 (for the case  $n = 3$ ). Each  $\Gamma_k$  contains part of  $C_k$  and part of  $\Gamma$ ;  $\Gamma'$  contains parts of  $\Gamma$ , and goes 'round the back' of each  $C_k$ .

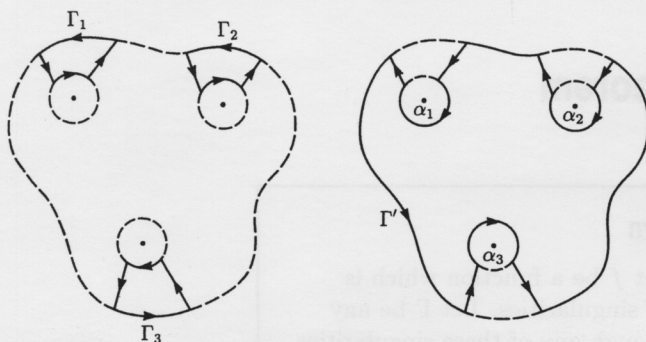


Figure 2.8

Note, for example, that  $\Gamma'$  contains  $\tilde{L}_1$  and  $L'_1$ , whereas  $\Gamma_1$  contains  $L_1$  and  $\tilde{L}'_1$ , the reverse contours of  $\tilde{L}_1$  and  $L'_1$ ; so the associated integrals cancel when they are added. Also note, for example, that the part of the circle round  $\alpha_1$  in  $\Gamma'$  and the part in  $\Gamma_1$  combine to give  $\tilde{C}_1$ .

Thus we have

$$\int_{\Gamma} f(z) dz = \int_{\Gamma'} f(z) dz + \sum_{k=1}^n \left( \int_{\Gamma_k} f(z) dz \right) + \sum_{k=1}^n \left( \int_{C_k} f(z) dz \right).$$

But, by Cauchy's Theorem (applied to  $\Gamma'$  and to each  $\Gamma_k$  on appropriate simply-connected regions),

$$\int_{\Gamma'} f(z) dz = 0 \quad \text{and} \quad \int_{\Gamma_k} f(z) dz = 0.$$

Thus

$$\begin{aligned} \int_{\Gamma} f(z) dz &= \sum_{k=1}^n \left( \int_{C_k} f(z) dz \right) \\ &= \sum_{k=1}^n 2\pi i \operatorname{Res}(f, \alpha_k) \quad (\text{by Equation (2.2)}) \\ &= 2\pi i \sum_{k=1}^n \operatorname{Res}(f, \alpha_k). \end{aligned}$$

This completes the proof. ■



### 3 EVALUATING IMPROPER INTEGRALS

After working through this section, you should be able to:

- (a) use the Residue Theorem to evaluate certain improper real integrals of the forms

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} dt, \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \cos kt dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \sin kt dt,$$

where  $p$  and  $q$  are polynomial functions.

#### 3.1 Improper integrals

Integrals of the forms

$$\int_{-\infty}^{\infty} f(t) dt \quad \text{and} \quad \int_0^{\infty} f(t) dt,$$

where  $f$  is a real-valued or complex-valued function of the real variable  $t$ , occur in many branches of mathematics. They are called *improper integrals*. For example, the improper integral

$$\int_{-\infty}^{\infty} e^{-t^2} dt$$

is of importance in statistics, because of its connection with the normal distribution, and the *Laplace transform* of a function  $f$ , defined by

$$L_f(z) = \int_0^{\infty} e^{-zt} f(t) dt,$$

arises in the theory of differential equations.

Our aim in this subsection is to define such improper integrals. The idea of the definition is to consider an ordinary integral such as

$$\int_0^r e^{-t} dt,$$

which represents the shaded area in Figure 3.1, and then *define*

$$\int_0^{\infty} e^{-t} dt = \lim_{r \rightarrow \infty} \int_0^r e^{-t} dt,$$

provided that this limit exists.

First, however, we need to define the notion of a limit as  $r$  tends to  $\infty$ .

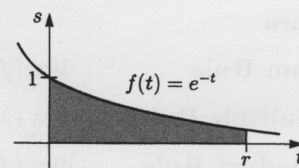


Figure 3.1

**Definition** Let  $f$  be a function defined on an unbounded interval  $]a, \infty[$ . Then the function  $f$  has limit  $\alpha$  as  $r$  tends to  $\infty$  if

for each real sequence  $\{r_n\}$  in  $]a, \infty[$  such that  $r_n \rightarrow \infty$ ,  
 $f(r_n) \rightarrow \alpha$  as  $n \rightarrow \infty$ .

In this case we write

EITHER  $\lim_{r \rightarrow \infty} f(r) = \alpha$ ,

OR  $f(r) \rightarrow \alpha$  as  $r \rightarrow \infty$ .

**Remark** There is an equivalent  $\varepsilon$ - $N$  definition of this limit in which the displayed condition is replaced by

for each positive number  $\varepsilon$ , there is an integer  $N$  such that  
 $|f(r) - \alpha| < \varepsilon$ , for all  $r > N$ .

### Example 3.1

Evaluate the following limit.

$$\lim_{r \rightarrow \infty} \frac{1}{r^2}$$

#### Solution

Intuitively, we suspect that the limit is 0. We show that this is the case.

First note that the function  $f(z) = 1/z^2$  is defined on  $]0, \infty[$ . If  $\{r_n\}$  is a sequence in  $]0, \infty[$  such that  $r_n \rightarrow \infty$ , then  $1/r_n \rightarrow 0$  by the Reciprocal Rule, and hence  $1/r_n^2 \rightarrow 0$ , by the Product Rule for sequences. Hence

Unit A3, Theorem 1.5

$$f(r) = 1/r^2 \rightarrow 0 \text{ as } r \rightarrow \infty. \quad \blacksquare$$

### Problem 3.1

Use the  $\varepsilon$ - $N$  definition to prove that

$$\lim_{r \rightarrow \infty} \frac{1}{\sqrt{r}} = 0.$$

More complicated limits can be evaluated by using the following Combination Rules, which follow from the Combination Rules for sequences. We omit their proofs.

#### Theorem 3.1 Combination Rules

Let  $f$  and  $g$  be functions such that

$$\lim_{r \rightarrow \infty} f(r) = \alpha \quad \text{and} \quad \lim_{r \rightarrow \infty} g(r) = \beta.$$

Then

**Sum Rule**  $\lim_{r \rightarrow \infty} (f(r) + g(r)) = \alpha + \beta;$

**Multiple Rule**  $\lim_{r \rightarrow \infty} (\lambda f(r)) = \lambda \alpha, \quad \text{for } \lambda \in \mathbb{C};$

**Product Rule**  $\lim_{r \rightarrow \infty} (f(r)g(r)) = \alpha\beta;$

**Quotient Rule**  $\lim_{r \rightarrow \infty} (f(r)/g(r)) = \alpha/\beta, \quad \text{provided that } \beta \neq 0.$

### Example 3.2

Evaluate the following limit.

$$\lim_{r \rightarrow \infty} \frac{\pi r}{r^2 - 1}$$

We shall need this limit later in this section.

#### Solution

It seems likely that the limit is 0, since the power of  $r$  occurring in the denominator is larger than that occurring in the numerator.

To prove this, we divide the numerator and denominator by  $r^2$ , and write

$r^2$  is the 'dominant term'.

$$\lim_{r \rightarrow \infty} \frac{\pi r}{r^2 - 1} = \lim_{r \rightarrow \infty} \frac{\pi/r}{1 - (1/r^2)}.$$

But

$$\lim_{r \rightarrow \infty} 1/r = 0 \quad \text{and} \quad \lim_{r \rightarrow \infty} 1/r^2 = 0.$$

It follows from the Sum, Multiple and Quotient Rules that

$$\lim_{r \rightarrow \infty} \frac{\pi/r}{1 - (1/r^2)} = \frac{\pi \cdot 0}{1 - 0} = 0. \quad \blacksquare$$



The limit evaluated in Example 3.2 is a special case of the following useful corollary to Theorem 3.1 (which we often use without explicit reference).

**Corollary** If  $p$  and  $q$  are polynomial functions such that the degree of  $q$  exceeds the degree of  $p$ , then

$$\lim_{r \rightarrow \infty} \frac{p(r)}{q(r)} = 0.$$

### Problem 3.2

Prove the corollary to Theorem 3.1.

We can now define the improper integrals

$$\int_{-\infty}^{\infty} f(t) dt \quad \text{and} \quad \int_a^{\infty} f(t) dt.$$

**Definitions** Let  $f$  be a continuous function with domain  $\mathbb{R}$ . Then the **improper integral**  $\int_{-\infty}^{\infty} f(t) dt$  is

$$\int_{-\infty}^{\infty} f(t) dt = \lim_{r \rightarrow \infty} \int_{-r}^r f(t) dt, \quad (3.1)$$

provided that this limit exists.

Let  $f$  be a function which is continuous on the interval  $[a, \infty[$ . Then the **improper integral**  $\int_a^{\infty} f(t) dt$  is

$$\int_a^{\infty} f(t) dt = \lim_{r \rightarrow \infty} \int_a^r f(t) dt, \quad (3.2)$$

provided that this limit exists.

### Example 3.3

Evaluate the following improper integrals.

(a)  $\int_{-\infty}^{\infty} t^3 dt$       (b)  $\int_0^{\infty} \frac{1}{t^2 + 1} dt$

**Solution**

$$\begin{aligned} \text{(a)} \quad \int_{-\infty}^{\infty} t^3 dt &= \lim_{r \rightarrow \infty} \int_{-r}^r t^3 dt \\ &= \lim_{r \rightarrow \infty} \left[ \frac{1}{4} t^4 \right]_{-r}^r \\ &= \lim_{r \rightarrow \infty} 0 = 0. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_0^{\infty} \frac{1}{t^2 + 1} dt &= \lim_{r \rightarrow \infty} \int_0^r \frac{1}{t^2 + 1} dt \\ &= \lim_{r \rightarrow \infty} [\tan^{-1} t]_0^r \\ &= \lim_{r \rightarrow \infty} \tan^{-1} r = \frac{1}{2} \pi. \quad \blacksquare \end{aligned}$$

**Remark** An alternative, but *not* equivalent, definition of the improper

integral  $\int_{-\infty}^{\infty} f(t) dt$  is

$$\int_{-\infty}^{\infty} f(t) dt = \int_{-\infty}^0 f(t) dt + \int_0^{\infty} f(t) dt, \quad (3.3)$$

where  $\int_0^{\infty} f(t) dt$  is defined as above and

$$\int_{-\infty}^0 f(t) dt = \lim_{r \rightarrow \infty} \int_{-r}^0 f(t) dt,$$

provided that this limit exists.

If the integrals on the right-hand side of Equation (3.3) exist, then the two definitions of  $\int_{-\infty}^{\infty} f(t) dt$  give the same value. However this does not always happen; for example, the integral in Example 3.3(a) defined according to Equation (3.3) does *not* exist, since

$$\int_0^r t^3 dt = \frac{1}{4} r^4 \rightarrow \infty \text{ as } r \rightarrow \infty,$$

and so  $\int_0^{\infty} t^3 dt$  does not exist. Under our definition,  $\int_{-\infty}^{\infty} t^3 dt$  exists because,

for each value of  $r$ ,  $\int_0^r t^3 dt$  (which represents the 'positive area' on the right in Figure 3.2) cancels  $\int_{-r}^0 t^3 dt$  (the 'negative area' on the left).

Texts that do not adopt our definition refer to our integral as the *principal value of the integral*, denoted, for example, by  $\text{PV} \int_{-\infty}^{\infty} t^3 dt = 0$ .

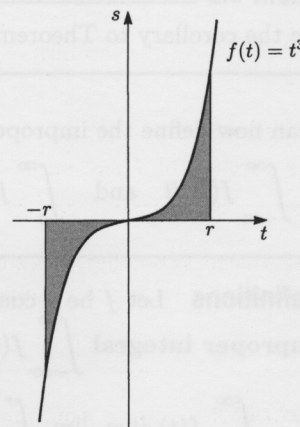


Figure 3.2

### Problem 3.3

Evaluate each of the following improper integrals.

(a)  $\int_{-\infty}^{\infty} \sin t dt$       (b)  $\int_1^{\infty} \frac{1}{t^p} dt$ , where  $p > 1$       (c)  $\int_0^{\infty} e^{-t} dt$

The following results are very useful when we wish to evaluate certain improper real integrals. Recall that, if  $f$  is a function with domain  $A$ , then

$f$  is *odd* if  $f(-t) = -f(t)$ , for all  $t \in A$ ;

$f$  is *even* if  $f(-t) = f(t)$ , for all  $t \in A$ .

For example, the function

$f(t) = t^3$  is odd,

and the function

$f(t) = 1/(t^2 + 1)$  is even.

**Theorem 3.2** Let  $f$  be a continuous function with domain  $\mathbb{R}$ . Then

(a) if  $f$  is an odd function,  $\int_{-\infty}^{\infty} f(t) dt = 0$ ;

(b) if  $f$  is an even function,  $\int_{-\infty}^{\infty} f(t) dt = 2 \int_0^{\infty} f(t) dt$ ,  
provided that these improper integrals exist.

We prove part (a), leaving you to prove part (b) in Problem 3.4 below.



## Proof

(a) If  $f$  is an odd function, then

$$\begin{aligned}
 \int_{-\infty}^{\infty} f(t) dt &= \lim_{r \rightarrow \infty} \int_{-r}^r f(t) dt \\
 &= \lim_{r \rightarrow \infty} \left( \int_0^r f(t) dt + \int_{-r}^0 f(t) dt \right) \\
 &= \lim_{r \rightarrow \infty} \left( \int_0^r f(t) dt + \int_r^0 f(-u)(-du) \right) \\
 &= \lim_{r \rightarrow \infty} \left( \int_0^r f(t) dt - \int_0^r f(u) du \right) \\
 &= \lim_{r \rightarrow \infty} 0 = 0. \quad \blacksquare
 \end{aligned}$$

$$t = -u, \quad dt = -du$$

$$f(-u) = -f(u)$$

## Problem 3.4

Prove part (b) of Theorem 3.2.

We also need two other types of improper integral. They both involve real functions which are continuous on some interval except at one point, such as  $f(t) = 1/t$ .

**Definitions** Let a function  $f$  be defined and continuous at all points of an interval  $[a, b]$  except at the point  $c \in ]a, b[$  (see Figure 3.3). Then the **improper integral**  $\int_a^b f(t) dt$  is

$$\int_a^b f(t) dt = \lim_{\varepsilon \rightarrow 0} \left( \int_a^{c-\varepsilon} f(t) dt + \int_{c+\varepsilon}^b f(t) dt \right),$$

where the limit is taken as  $\varepsilon$  tends to 0 through *positive* values, provided that this limit exists.

Let a function  $f$  be continuous at all points of  $\mathbb{R}$ , except at the point  $c$ .

Then the **improper integral**  $\int_{-\infty}^{\infty} f(t) dt$  is

$$\begin{aligned}
 \int_{-\infty}^{\infty} f(t) dt &= \lim_{r \rightarrow \infty} \int_{-r}^r f(t) dt \\
 &= \lim_{r \rightarrow \infty} \left( \lim_{\varepsilon \rightarrow 0} \left( \int_{-r}^{c-\varepsilon} f(t) dt + \int_{c+\varepsilon}^r f(t) dt \right) \right),
 \end{aligned}$$

provided that these limits exist.

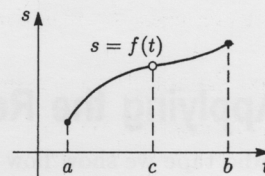


Figure 3.3

**Remark** The above definitions can easily be extended to functions which are continuous except at a finite number of points.

### Example 3.4

Evaluate the improper integral  $\int_{-\infty}^{\infty} \frac{1}{t^3} dt$ .

#### Solution

The function  $f(t) = 1/t^3$  is continuous at all points of  $\mathbb{R}$ , except at the point 0 (see Figure 3.4). Thus, for  $r > 0$ ,

$$\begin{aligned}\int_{-r}^r \frac{1}{t^3} dt &= \lim_{\varepsilon \rightarrow 0} \left( \int_{-r}^{-\varepsilon} \frac{1}{t^3} dt + \int_{\varepsilon}^r \frac{1}{t^3} dt \right) \\ &= \lim_{\varepsilon \rightarrow 0} \left( \left[ -\frac{1}{2t^2} \right]_{-r}^{-\varepsilon} + \left[ -\frac{1}{2t^2} \right]_{\varepsilon}^r \right) \\ &= \lim_{\varepsilon \rightarrow 0} \left( -\frac{1}{2\varepsilon^2} + \frac{1}{2r^2} - \frac{1}{2r^2} + \frac{1}{2\varepsilon^2} \right) \\ &= \lim_{\varepsilon \rightarrow 0} 0 = 0,\end{aligned}$$

and hence

$$\int_{-\infty}^{\infty} \frac{1}{t^3} dt = \lim_{r \rightarrow \infty} \int_{-r}^r \frac{1}{t^3} dt = \lim_{r \rightarrow \infty} 0 = 0. \quad \blacksquare$$

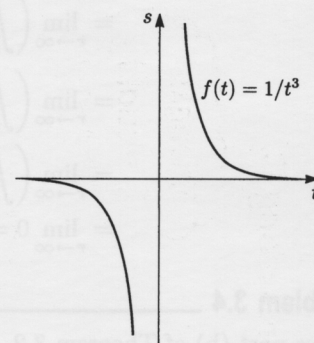


Figure 3.4

This answer is to be expected because  $f(t) = 1/t^3$  is an odd function.

### Problem 3.5

Evaluate the improper integral  $\int_{-1}^2 \frac{1}{t} dt$ .

## 3.2 Applying the Residue Theorem (audio-tape)

In the audio tape we show how the Residue Theorem can be used to evaluate a whole class of real improper integrals of the forms

$$\int_{-\infty}^{\infty} f(t) dt, \quad \int_{-\infty}^{\infty} f(t) \cos kt dt, \quad \text{and} \quad \int_{-\infty}^{\infty} f(t) \sin kt dt,$$

where  $f$  is a real rational function.

In Frame 1 we start with a straightforward example and then consider more involved ones in Frames 2 and 3.

NOW START THE TAPE.

Recall that  $f$  is a real rational function if

$$f(t) = \frac{p(t)}{q(t)},$$

where  $p$  and  $q$  are real polynomial functions.





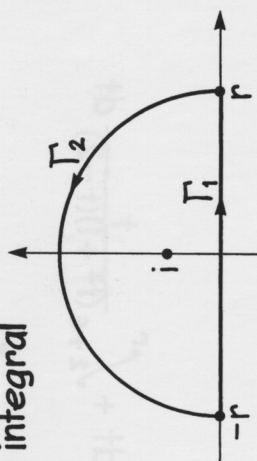
# 1. Evaluate $\int_{-\infty}^{\infty} \frac{1}{t^2+1} dt$

$$= \lim_{r \rightarrow \infty} \int_{-r}^r \frac{1}{t^2+1} dt$$

(a) Consider the contour integral

$$I = \int_{\Gamma} \frac{1}{z^2+1} dz,$$

where  $\Gamma = \Gamma_1 + \Gamma_2$ .



(b) Use the Residue Theorem

$$I = 2\pi i \times (\text{residue at } i)$$

$$= 2\pi i \times \frac{1}{2i}$$

$$= \pi$$

g/h Rule:

$$g(z)=1, h(z)=z^2+1$$

independent of  $r > 1$

(c) Split up the integral

$$I = \int_{\Gamma_1} \frac{1}{z^2+1} dz + \int_{\Gamma_2} \frac{1}{z^2+1} dz$$

$$= \int_{-r}^r \frac{1}{t^2+1} dt + \int_{\Gamma_2} \frac{1}{z^2+1} dz$$

(\*)

$$= \pi, \text{ by (b)}$$

(d) Estimate the integral along  $\Gamma_2$

On  $\Gamma_2$ ,  $|z|=r$ , so

$$|z^2+1| \geq r^2-1;$$

hence

$$\left| \frac{1}{z^2+1} \right| \leq \frac{1}{r^2-1}.$$

$$r > 1$$

Thus

$$\left| \int_{\Gamma_2} \frac{1}{z^2+1} dz \right| \leq \frac{\pi r}{r^2-1}.$$

Estimation Theorem

$$M = 1/(r^2-1)$$

$$L = \pi r$$

(e) Let  $r \rightarrow \infty$  in (\*)

$$\int_{-r}^r \frac{1}{t^2+1} dt + \int_{\Gamma_2} \frac{1}{z^2+1} dz = \pi$$

$$\rightarrow \int_{-\infty}^{\infty} ?$$

$$\lim_{r \rightarrow \infty} \int_{\Gamma_2} \frac{1}{z^2+1} dz = 0,$$

by (d)

Hence

$$\int_{-\infty}^{\infty} \frac{1}{t^2+1} dt = \lim_{r \rightarrow \infty} \int_{-r}^r \frac{1}{t^2+1} dt$$

$$= \pi.$$

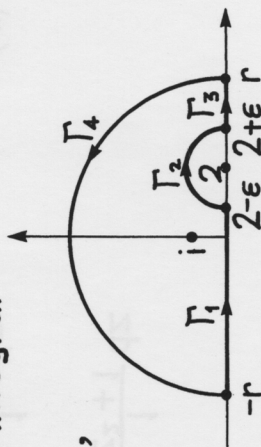
Now try  
Problem 3.6.

$$2. \text{ Evaluate } \int_{-\infty}^{\infty} \frac{t}{(t^2+1)(t-2)} dt$$

(a) Consider the contour integral

$$I = \int_{\Gamma} \frac{z}{(z^2+1)(z-2)} dz,$$

where  $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ .



(b) Use the Residue Theorem

$$I = 2\pi i \times (\text{residue at } i)$$

$$= 2\pi i \times \left( \frac{i/(i-2)}{2i} \right)$$

$$= \frac{\pi}{5} - \frac{2\pi i}{5}$$

(c) Split up the integral

$$I = \int_{-r}^{2-\epsilon} \frac{t}{(t^2+1)(t-2)} dt + \int_{2+\epsilon}^r \frac{t}{(t^2+1)(t-2)} dt$$

$$+ \int_{\Gamma_2} \frac{z}{(z^2+1)(z-2)} dz + \int_{\Gamma_4} \frac{z}{(z^2+1)(z-2)} dz \quad (*)$$

$$= \frac{\pi}{5} - \frac{2\pi i}{5}, \text{ by (b)}$$

(d) Estimate the integral along  $\Gamma_4$

On  $\Gamma_4$ ,  $|z| = r$ , so

$$|z^2+1| \geq r^2-1 \text{ and } |z-2| \geq r-2;$$

hence

$$\left| \frac{z}{(z^2+1)(z-2)} \right| \leq \frac{r}{(r^2-1)(r-2)}.$$

Thus

$$\left| \int_{\Gamma_4} \frac{z}{(z^2+1)(z-2)} dz \right| \leq \frac{\pi r^2}{(r^2-1)(r-2)}.$$

(e) Let  $\epsilon \rightarrow 0$  and  $r \rightarrow \infty$  in (\*)

$$\int_{-r}^{2-\epsilon} \dots dt + \int_{2+\epsilon}^r \dots dt + \int_{\Gamma_2} \dots dz + \int_{\Gamma_4} \dots dz = \frac{\pi}{5} - \frac{2\pi i}{5}$$

$$\rightarrow \int_{-\infty}^{\infty} ?$$

$$\lim_{r \rightarrow \infty} \int_{\Gamma_4} \dots dz = 0, \text{ by (d)}$$

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_2} \dots dz = -\pi i \times (\text{residue at } 2)$$

$$= -\pi i \times \frac{2}{5} = -\frac{2\pi i}{5}$$

Round-the-Pole Lemma

Cover-up Rule

Hence

$$\int_{-\infty}^{\infty} \frac{t}{(t^2+1)(t-2)} dt = \left( \frac{\pi}{5} - \frac{2\pi i}{5} \right) - \left( -\frac{2\pi i}{5} \right) = \frac{\pi}{5}.$$

Now try Problem 3.7.

Triangle Inequality

$r > 2$

$M$

$L = \pi r$



### 3. Evaluate $\int_{-\infty}^{\infty} \frac{\cos t}{t^2 + 4} dt$

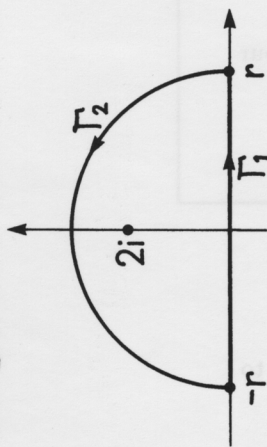
$$= \lim_{r \rightarrow \infty} \int_{-r}^r \frac{\cos t}{t^2 + 4} dt$$

(a) Consider the contour integral

$$I = \int_{\Gamma} \frac{e^{iz}}{z^2 + 4} dz,$$

where  $\Gamma = \Gamma_1 + \Gamma_2$ .

$$e^{it} = \cos t + i \sin t$$



(b) Use the Residue Theorem

$$I = 2\pi i \times (\text{residue at } 2i)$$

$$= 2\pi i \times \frac{e^{-2}}{4i}$$

$$= \frac{\pi}{2} e^2 \leftarrow \text{independent of } r > 2$$

g/h Rule:

$$g(z) = e^{iz},$$

$$h(z) = z^2 + 4$$

(c) Split up the integral

$$I = \int_{-r}^r \frac{e^{it}}{t^2 + 4} dt + \int_{\Gamma_2} \frac{e^{iz}}{z^2 + 4} dz$$

$$= \int_{-r}^r \frac{\cos t}{t^2 + 4} dt + i \int_{-r}^r \frac{\sin t}{t^2 + 4} dt + \int_{\Gamma_2} \frac{e^{iz}}{z^2 + 4} dz \quad (*)$$

$$= \pi/2 e^2, \text{ by (b)}$$

(d) Estimate the integral along  $\Gamma_2$

On  $\Gamma_2$ ,  $|z| = r$ , so

$$\left| \frac{1}{z^2 + 4} \right| \leq \frac{1}{r^2 - 4},$$

and

$$|e^{iz}| = |e^{ix} e^{-y}| = e^{-y} \leq 1.$$

Thus

$$\left| \int_{\Gamma_2} \frac{e^{iz}}{z^2 + 4} dz \right| \leq \frac{\pi r}{r^2 - 4}.$$

(e) Let  $r \rightarrow \infty$  in (\*)

$$\int_{-r}^r \frac{\cos t}{t^2 + 4} dt + i \int_{-r}^r \frac{\sin t}{t^2 + 4} dt + \int_{\Gamma_2} \frac{e^{iz}}{z^2 + 4} dz = \frac{\pi}{2} e^2$$

$$\rightarrow \int_{-\infty}^{\infty} ?$$

$$\rightarrow i \int_{-\infty}^{\infty} ?$$

$$\lim_{r \rightarrow \infty} \int_{\Gamma_2} \dots dz = 0, \text{ by (d)}$$

Hence

$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2 + 4} dt = \frac{\pi}{2} e^2$$

and

$$\int_{-\infty}^{\infty} \frac{\sin t}{t^2 + 4} dt = 0 \text{ (as expected).}$$

Now try Problem 3.8.

Triangle Inequality

$$r > 2$$

$$y \geq 0 \text{ on } \Gamma_2$$

$$M = 1/(r^2 - 4)$$

$$L = \pi r$$

Equate real and imaginary parts.

**Problem 3.6**

Use the method in Frame 1 to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{t^4 + 1} dt.$$

**Problem 3.7**

Use the method in Frame 2 to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{t}{t^3 - 1} dt.$$

**Problem 3.8**

Use the method in Frames 2 and 3 to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\sin 2t}{t^3 + 1} dt.$$

### 3.3 Some general results

In the audio tape we used the Residue Theorem to evaluate several improper integrals. In this subsection we justify some of the results used in the audio tape, and give two general results which contain formulas for such improper integrals under appropriate conditions.

First we state and prove the Round-the-Pole Lemma, which provides the justification for a result used in Frame 2.

**Lemma 3.1 Round-the-Pole Lemma**

If a function  $f$  has a simple pole at  $\alpha$  and  $\Gamma$  is the semicircular contour (traversed anticlockwise) from  $\alpha + \varepsilon$  to  $\alpha - \varepsilon$ , then

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma} f(z) dz = \pi i \operatorname{Res}(f, \alpha).$$

In Frame 2, we needed to evaluate

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_2} \frac{z}{(z^2 + 1)(z - 2)} dz,$$

where  $\Gamma_2$  is the semicircular contour in the upper half-plane from  $2 - \varepsilon$  to  $2 + \varepsilon$ . Denoting the integrand by  $f(z)$ , we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_2} f(z) dz &= \lim_{\varepsilon \rightarrow 0} \left( - \int_{\bar{\Gamma}_2} f(z) dz \right) \\ &= -\pi i \cdot \operatorname{Res}(f, 2), \end{aligned}$$

by the Round-the-Pole Lemma, as stated in Frame 2.

**Proof** Since  $f$  has a simple pole at  $\alpha$ , its Laurent series about  $\alpha$  is

$$\begin{aligned} f(z) &= \frac{a_{-1}}{z - \alpha} + a_0 + a_1(z - \alpha) + \cdots \\ &= \frac{\operatorname{Res}(f, \alpha)}{z - \alpha} + g(z), \end{aligned}$$

where the function  $g$  is analytic on the open disc  $D = \{z : |z - \alpha| < \delta\}$ , say. Thus, for  $0 < \varepsilon < \delta$ , we have

$$\int_{\Gamma} f(z) dz = \operatorname{Res}(f, \alpha) \int_{\Gamma} \frac{1}{z - \alpha} dz + \int_{\Gamma} g(z) dz.$$



Since  $g$  is analytic on  $D$ , there is a number  $M$  such that

$$|g(z)| \leq M, \quad \text{for } |z - \alpha| \leq \frac{1}{2}\delta,$$

and so, by the Estimation Theorem,

$$\left| \int_{\Gamma} g(z) dz \right| \leq M \times \pi\varepsilon, \quad \text{for } 0 < \varepsilon \leq \frac{1}{2}\delta.$$

Unit A3, Theorem 5.3

$$L = \pi\varepsilon$$

Hence

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma} g(z) dz = 0.$$

Now, using the (standard) parametrization  $\gamma(t) = \alpha + \varepsilon e^{it}$  ( $t \in [0, \pi]$ ) for  $\Gamma$ , we have

$$\int_{\Gamma} \frac{1}{z - \alpha} dz = \int_0^{\pi} \frac{\varepsilon i e^{it}}{\varepsilon e^{it}} dt = \pi i,$$

and so

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma} f(z) dz = \pi i \operatorname{Res}(f, \alpha),$$

as required. ■

Our first general result can be used to evaluate integrals such as those in Frames 1 and 2.

**Theorem 3.3** Let  $p$  and  $q$  be polynomial functions such that:

1. the degree of  $q$  exceeds that of  $p$  by at least two;
2. any poles of  $p/q$  on the real axis are simple.

Then

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} dt = 2\pi i S + \pi i T,$$

where  $S$  is the sum of the residues of the function  $p/q$  at those poles in the upper half-plane, and

$T$  is the sum of the residues of the function  $p/q$  at those poles on the real axis.

For example, in Frame 2, the integral is

$$\int_{-\infty}^{\infty} \frac{t}{(t^2 + 1)(t - 2)} dt,$$

so that  $p(z) = z$  and  $q(z) = (z^2 + 1)(z - 2)$ .

The only pole of  $p/q$  in the upper half-plane is at  $i$ , and the only pole (a simple one) of  $p/q$  on the real axis is at 2.

Hence

$$S = \operatorname{Res}(p/q, i) = \frac{i/(i - 2)}{2i}, \quad \text{from Frame 2,}$$

and

$$T = \operatorname{Res}(p/q, 2) = \frac{2}{5}, \quad \text{from Frame 2.}$$

Hence, by Theorem 3.3,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{t}{(t^2 + 1)(t - 2)} dt &= 2\pi i \left( \frac{i/(i - 2)}{2i} \right) + \pi i \left( \frac{2}{5} \right) \\ &= \frac{-\pi i(i + 2)}{5} + \frac{2\pi i}{5} \\ &= \frac{\pi}{5}, \quad \text{as in Frame 2.} \end{aligned}$$

Before reading the proof of Theorem 3.3, you should attempt the following problems.

### Problem 3.9

Use Theorem 3.3 to evaluate

$$\int_{-\infty}^{\infty} \frac{1}{t(t-1)(t^2+1)} dt.$$

### Problem 3.10

Use Theorem 3.3 to show that if  $a, b > 0$  and  $a \neq b$ , then

$$\int_{-\infty}^{\infty} \frac{1}{(t^2+a^2)(t^2+b^2)} dt = \frac{\pi}{ab(a+b)}.$$

### Proof of Theorem 3.3

We follow the five steps used in Frames 1 and 2 of the audio tape.

(a) First consider the contour integral

$$I = \int_{\Gamma} \frac{p(z)}{q(z)} dz,$$

where the contour  $\Gamma$  is shown in Figure 3.5, chosen to contain all the poles  $\alpha_1, \alpha_2, \dots, \alpha_k$  of  $p/q$  in the upper half-plane, but to exclude those (simple) poles  $\beta_1, \beta_2, \dots, \beta_\ell$  of  $p/q$  on the real axis.

(b) By the Residue Theorem,

$$I = 2\pi i S.$$

(c) Now we split up the integral  $I$  as follows:

$$I = \int_{\Gamma_1} \frac{p}{q} + \int_{\Gamma_2} \frac{p}{q} + \dots + \int_{\Gamma_{2\ell-1}} \frac{p}{q} + \int_{\Gamma_{2\ell}} \frac{p}{q} + \int_{\Gamma_{2\ell+1}} \frac{p}{q} + \int_{\Gamma_{2\ell+2}} \frac{p}{q},$$

where the contours  $\Gamma_1, \Gamma_2, \dots, \Gamma_{2\ell+2}$  are as shown in Figure 3.6.

(d) To estimate the integral along the large semicircle  $\Gamma_{2\ell+2}$ , we suppose that

$$p(z) = a_0 + a_1 z + \dots + a_m z^m,$$

$$q(z) = b_0 + b_1 z + \dots + b_n z^n,$$

where  $a_m \neq 0, b_n \neq 0$  and  $n \geq m+2$ . Now, let

$$\begin{aligned} M(r) &= \max\{|p(z)/q(z)| : |z| = r\} \\ &= \max\{|p(z)|/|q(z)| : |z| = r\}. \end{aligned}$$

By the Triangle Inequality,

$$|p(z)| \leq |a_0| + |a_1|r + \dots + |a_m|r^m, \quad \text{for } |z| = r,$$

and

$$|q(z)| \geq ||b_n|r^n - |b_{n-1}|r^{n-1} - \dots - |b_1|r - |b_0||, \quad \text{for } |z| = r.$$

Hence

$$M(r) \leq \frac{|a_0| + |a_1|r + \dots + |a_m|r^m}{||b_n|r^n - |b_{n-1}|r^{n-1} - \dots - |b_1|r - |b_0||},$$

so that

$$rM(r) \rightarrow 0 \text{ as } r \rightarrow \infty,$$

since  $n > m+1$ . By the Estimation Theorem,

$$\left| \int_{\Gamma_{2\ell+2}} \frac{p(z)}{q(z)} dz \right| \leq M(r) \times \pi r,$$

This proof may be omitted on a first reading.

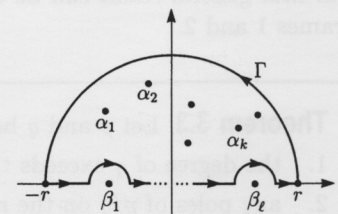


Figure 3.5

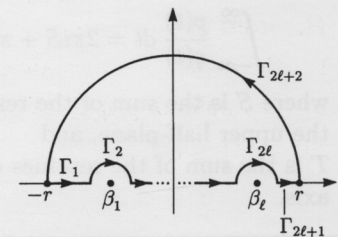


Figure 3.6

$$\begin{aligned} M &= M(r), \\ L &= \pi r. \end{aligned}$$



and so

$$\lim_{r \rightarrow \infty} \int_{\Gamma_{2\ell+2}} \frac{p(z)}{q(z)} dz = 0.$$

(e) By the Round-the-Pole Lemma, we have

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_{2j}} \frac{p(z)}{q(z)} dz = -\pi i \operatorname{Res}(p/q, \beta_j), \quad \text{for } j = 1, 2, \dots, \ell,$$

since each  $\Gamma_{2j}$  is traversed clockwise. Thus

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_1 + \Gamma_2 + \dots + \Gamma_{2\ell+1}} \frac{p}{q} &= \lim_{\epsilon \rightarrow 0} \left( \int_{\Gamma_1} \frac{p}{q} + \int_{\Gamma_3} \frac{p}{q} + \dots + \int_{\Gamma_{2\ell+1}} \frac{p}{q} \right) \\ &\quad + \lim_{\epsilon \rightarrow 0} \left( \int_{\Gamma_2} \frac{p}{q} + \int_{\Gamma_4} \frac{p}{q} + \dots + \int_{\Gamma_{2\ell}} \frac{p}{q} \right) \\ &= \int_{-r}^r \frac{p(t)}{q(t)} dt - \pi i T. \end{aligned}$$

By parts (b) and (c), therefore,

$$2\pi i S = \int_{-r}^r \frac{p(t)}{q(t)} dt - \pi i T + \int_{\Gamma_{2\ell+2}} \frac{p(z)}{q(z)} dz,$$

and hence, by part (d),

$$\lim_{r \rightarrow \infty} \int_{-r}^r \frac{p(t)}{q(t)} dt = 2\pi i S + \pi i T,$$

as required. ■

Our next general result can be used to evaluate improper integrals such as the one in Frame 3 of the audio tape.

**Theorem 3.4** Let  $p$  and  $q$  be polynomial functions such that:

1. the degree of  $q$  exceeds that of  $p$  by at least one;
2. any poles of  $p/q$  on the real axis are simple.

Then, if  $k > 0$ ,

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} e^{ikt} dt = 2\pi i S + \pi i T, \quad (3.4)$$

where  $S$  is the sum of the residues of the function  $z \mapsto (p(z)/q(z))e^{ikz}$  at those poles in the upper half-plane, and

$T$  is the sum of the residues of the function  $z \mapsto (p(z)/q(z))e^{ikz}$  at those poles on the real axis.

By equating the real parts and imaginary parts of Equation (3.4), we obtain the values of the real improper integrals

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \cos kt dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \sin kt dt.$$

The following problems will give you some practice at this technique.

### Problem 3.11

Use Theorem 3.4 to show that if  $k > 0$ , then

$$\int_{-\infty}^{\infty} \frac{e^{ikt}}{t} dt = i\pi.$$

Hence determine

$$\int_{-\infty}^{\infty} \frac{\cos kt}{t} dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin kt}{t} dt.$$

### Problem 3.12

Use Theorem 3.4 to determine

$$\int_{-\infty}^{\infty} \frac{\cos 2t}{t^2 + 9} dt.$$

The proof of Theorem 3.4 is similar to that of Theorem 3.3. Indeed, the proof would be *very* similar if Condition 1 of Theorem 3.4 were identical to Condition 1 of Theorem 3.3. However, the appearance of the exponential term  $e^{ikt}$  makes it possible to reduce the number 'two' to 'one' in Condition 1. The key to this reduction lies in the following result.

#### Lemma 3.2 Jordan's Lemma

Let  $\Gamma$  be the semicircular contour (traversed anticlockwise) from  $r$  to  $-r$ , and suppose that a function  $f$  is continuous on  $\Gamma$  and satisfies

$$|f(z)| \leq M, \quad \text{for } z \in \Gamma.$$

Then, for  $k > 0$ , we have

$$\left| \int_{\Gamma} f(z) e^{ikz} dz \right| \leq \frac{M\pi}{k}.$$

This is the same Jordan as in the Jordan Curve Theorem, Unit B2.

Now, under Condition 1 of Theorem 3.4,

$$M(r) = \max\{|p(z)/q(z)| : |z| = r\} \rightarrow 0 \text{ as } r \rightarrow \infty,$$

Corollary to Theorem 3.1

and so, by Jordan's Lemma,

$$\lim_{r \rightarrow \infty} \left| \int_{\Gamma} \frac{p(z)}{q(z)} e^{ikz} dz \right| \leq \lim_{r \rightarrow \infty} \frac{M(r)\pi}{k} = 0.$$

Hence the analogue of step (d) in the proof of Theorem 3.3 can be justified under Condition 1 of Theorem 3.4; the rest of the proof of Theorem 3.4 is almost identical to that of Theorem 3.3.

#### Proof of Jordan's Lemma

Using the standard parametrization  $\gamma(t) = re^{it}$  ( $t \in [0, \pi]$ ) for  $\Gamma$ , we have

This proof may be omitted on a first reading.

$$\begin{aligned} \int_{\Gamma} f(z) e^{ikz} dz &= \int_0^{\pi} f(re^{it}) \exp(ikre^{it}) ire^{it} dt \\ &= \int_0^{\pi} f(re^{it}) e^{ikr \cos t} e^{-kr \sin t} ire^{it} dt. \end{aligned}$$

$$\gamma'(t) = ire^{it}$$

$$\operatorname{Re}(ikre^{it}) = -kr \sin t$$

Hence, by Lemma 4.1 of Unit B1,

$$\begin{aligned} \left| \int_{\Gamma} f(z) e^{ikz} dz \right| &\leq \int_0^{\pi} |f(re^{it})| |e^{ikr \cos t}| |e^{-kr \sin t}| |ire^{it}| dt \\ &\leq Mr \int_0^{\pi} e^{-kr \sin t} dt \\ &= 2Mr \int_0^{\pi/2} e^{-kr \sin t} dt, \end{aligned} \quad (3.5)$$

$$\begin{aligned} |f(re^{it})| &\leq M, \\ |e^{ikr \cos t}| &= 1, \\ |ire^{it}| &= r. \end{aligned}$$

since the sine function is symmetric under reflection in the line  $t = \pi/2$  (that is,  $\sin(\pi - t) = \sin t$ ).



Now

$$\sin t \geq 2t/\pi, \quad \text{for } 0 \leq t \leq \pi/2,$$

(see Figure 3.7), and so

$$\begin{aligned} \int_0^{\pi/2} e^{-kr \sin t} dt &\leq \int_0^{\pi/2} e^{-2krt/\pi} dt \\ &= \left[ -\frac{\pi}{2kr} e^{-2krt/\pi} \right]_0^{\pi/2} \\ &= \frac{\pi}{2kr} (1 - e^{-kr}) \\ &< \frac{\pi}{2kr}. \end{aligned} \quad (3.6)$$

Combining Inequalities (3.5) and (3.6), we obtain

$$\left| \int_{\Gamma} f(z) e^{ikz} dz \right| \leq 2Mr \cdot \frac{\pi}{2kr} = \frac{M\pi}{k},$$

as required. ■

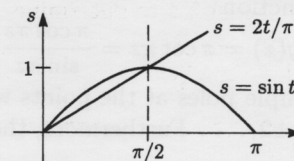


Figure 3.7

## 4 SUMMING SERIES

After working through this section, you should be able to:

(a) use the Residue Theorem to sum certain series of the forms

$$\sum_{n=1}^{\infty} \phi(n) \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n \phi(n),$$

where  $\phi$  is an appropriate even function.

### 4.1 Series of the form $\sum_{n=1}^{\infty} \phi(n)$

In this subsection, we shall use the theory of residues to sum certain real infinite series of the form

$$\sum_{n=1}^{\infty} \phi(n),$$

where  $\phi$  is an even function. In particular, we shall evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \dots,$$

which correspond to the even functions  $\phi(z) = z^{-2}$  and  $\phi(z) = z^{-4}$ .

At first sight, it may seem rather strange that a method for evaluating complex integrals can also be useful for summing real series. However, since we can integrate a function  $f$  along a simple-closed contour  $\Gamma$  by calculating

$$2\pi i \times (\text{the sum of the residues of } f \text{ at singularities inside } \Gamma),$$

it seems at least possible that we can make each such residue a term of the series to be summed. We may then be able to find the sum of the series by evaluating the corresponding integrals.

Recall that  $\phi: A \rightarrow \mathbb{C}$  is an even function if  $\phi(z) = \phi(-z)$ , for all  $z \in A$ .

It turns out that this is easily done. Remember first that the singularities of the function

$$g(z) = \pi \cot \pi z = \frac{\pi \cos \pi z}{\sin \pi z}$$

are simple poles at the points where  $\sin \pi z = 0$ , that is, at the integers  $0, \pm 1, \pm 2, \dots$ . Furthermore, the residue of  $g$  at each of these simple poles is 1. See Example 1.5(a).

To see why this is relevant, consider the function

$$f(z) = \pi \cot \pi z \cdot \phi(z),$$

where  $\phi$  is an even function analytic on  $\mathbb{C}$  apart from a finite number of poles, none of which occurs at an integer, except (possibly) at 0. Then the residue of  $f$  at a non-zero integer  $n$  is

$$\begin{aligned} \lim_{z \rightarrow n} (z - n) f(z) &= \lim_{z \rightarrow n} ((z - n) \pi \cot \pi z) \lim_{z \rightarrow n} \phi(z) \\ &= \text{Res}(g, n) \times \phi(n) \\ &= 1 \times \phi(n) = \phi(n). \end{aligned}$$

Thus the sum of the residues of  $f$  at the *positive* integers is

$$\phi(1) + \phi(2) + \dots = \sum_{n=1}^{\infty} \phi(n),$$

a sum of the type which we wish to evaluate. Note also that the sum of the residues at the *negative* integers is

$$\phi(-1) + \phi(-2) + \dots = \sum_{n=1}^{\infty} \phi(-n) = \sum_{n=1}^{\infty} \phi(n),$$

since  $\phi$  is an even function.

The method we use is to integrate the function  $f(z) = \pi \cot \pi z \cdot \phi(z)$  around the square contour  $S_N$  with vertices at the points

$$(N + \tfrac{1}{2})(1 + i), (N + \tfrac{1}{2})(-1 + i), (N + \tfrac{1}{2})(-1 - i), (N + \tfrac{1}{2})(1 - i),$$

where  $N$  is an integer large enough for  $S_N$  to contain all the non-zero poles  $\alpha_1, \alpha_2, \dots, \alpha_k$  of  $\phi$  (see Figure 4.1). This is possible, since there are only finitely many such poles. It follows from the Residue Theorem, applied to the function  $f(z) = \pi \cot \pi z \cdot \phi(z)$  and the contour  $S_N$ , that

$$\begin{aligned} \int_{S_N} f(z) dz &= \int_{S_N} \pi \cot \pi z \cdot \phi(z) dz \\ &= 2\pi i \left( \sum_{n=1}^N \text{Res}(f, n) + \sum_{n=1}^N \text{Res}(f, -n) + \text{Res}(f, 0) + \sum_{j=1}^k \text{Res}(f, \alpha_j) \right) \\ &= 2\pi i \left( \sum_{n=1}^N \phi(n) + \sum_{n=1}^N \phi(-n) + \text{Res}(f, 0) + \sum_{j=1}^k \text{Res}(f, \alpha_j) \right) \\ &= 2\pi i \left( 2 \sum_{n=1}^N \phi(n) + \text{Res}(f, 0) + \sum_{j=1}^k \text{Res}(f, \alpha_j) \right), \end{aligned}$$

since  $\phi$  is an even function.

We now let  $N$  tend to  $\infty$ . If the function  $\phi$  has been chosen so that the integral on the left tends to 0, that is,

$$\lim_{N \rightarrow \infty} \int_{S_N} \pi \cot \pi z \cdot \phi(z) dz = 0,$$

then we obtain

$$0 = 2\pi i \left( 2 \sum_{n=1}^{\infty} \phi(n) + \text{Res}(f, 0) + \sum_{j=1}^k \text{Res}(f, \alpha_j) \right).$$

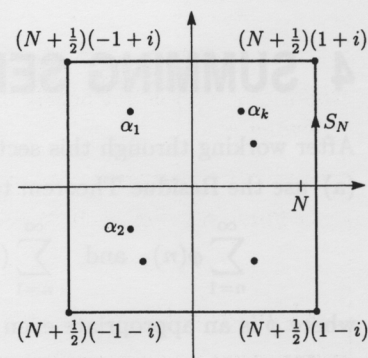


Figure 4.1



Rearranging this, we obtain

$$\sum_{n=1}^{\infty} \phi(n) = -\frac{1}{2} \left( \operatorname{Res}(f, 0) + \sum_{j=1}^k \operatorname{Res}(f, \alpha_j) \right).$$

Thus we have obtained the sum of the series in terms of the residues of  $f$  at 0 and at the non-zero poles of  $\phi$ .

We summarize this result in the following theorem.

**Theorem 4.1** Let  $\phi$  be an even function which is analytic on  $\mathbb{C}$  except for poles at the points  $\alpha_1, \alpha_2, \dots, \alpha_k$  (none of which is an integer), and possibly at 0, and let  $S_N$  be the square contour with vertices at  $(N + \frac{1}{2})(\pm 1 \pm i)$ . Suppose also that the function  $f(z) = \pi \cot \pi z \cdot \phi(z)$  is such that

$$\lim_{N \rightarrow \infty} \int_{S_N} f(z) dz = 0. \quad (4.1)$$

Then

$$\sum_{n=1}^{\infty} \phi(n) = -\frac{1}{2} \left( \operatorname{Res}(f, 0) + \sum_{j=1}^k \operatorname{Res}(f, \alpha_j) \right).$$

Condition (4.1) is not nearly as artificial as it may seem, in view of the following result.

**Lemma 4.1** For each  $N = 1, 2, \dots$ ,

$$|\cot \pi z| \leq 2, \quad \text{for } z \in S_N,$$

where  $S_N$  is the square contour with vertices at  $(N + \frac{1}{2})(\pm 1 \pm i)$ .

We prove this result at the end of the subsection.

Using this result, we can prove that Condition (4.1) holds for a wide variety of functions  $\phi$ . The reason we chose a square contour  $S_N$  instead of (say) a circular one is that the proof of Lemma 4.1 is simpler for square contours.

Let us now use Theorem 4.1 to sum a particular series.

### Example 4.1

Prove that  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ .

### Solution

The function  $\phi(z) = 1/z^2$  is even and analytic on  $\mathbb{C}$  apart from a pole of order two at 0.

The function

$$f(z) = (\pi \cot \pi z)/z^2$$

has a pole of order three at 0. By Theorem 1.2 with  $k = 3$ ,

$$\begin{aligned} \operatorname{Res}(f, 0) &= \frac{1}{2!} \lim_{z \rightarrow 0} \left( \frac{d^2}{dz^2} (\pi z \cot \pi z) \right) \\ &= -\frac{\pi^2}{3} \quad (\text{after some algebra}). \end{aligned}$$

We now check Condition (4.1). If  $z$  lies on the contour  $S_N$ , then

$$|z| \geq N + \frac{1}{2},$$

and so, by Lemma 4.1,

$$|f(z)| = \left| \frac{\pi \cot \pi z}{z^2} \right| \leq \frac{2\pi}{(N + \frac{1}{2})^2}, \quad \text{for } z \in S_N.$$

Hence, by the Estimation Theorem,

$$\left| \int_{S_N} f(z) dz \right| \leq \frac{2\pi}{(N + \frac{1}{2})^2} \cdot 4(2N + 1) = \frac{32\pi}{2N + 1},$$

$$M = \frac{2\pi}{(N + \frac{1}{2})^2},$$

$$L = 4(2N + 1).$$

which tends to 0 as  $N \rightarrow \infty$ . Thus Condition (4.1) holds.

It follows from Theorem 4.1 that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2} \text{Res}(f, 0) = -\frac{1}{2} \left( -\frac{\pi^2}{3} \right) = \frac{\pi^2}{6}. \quad \blacksquare$$

It is clear from the solution to Example 4.1 that a key step in evaluating the

sum  $\sum_{n=1}^{\infty} \phi(n)$  is the determination of the residue of the function

$$f(z) = \pi \cot \pi z \cdot \phi(z)$$

at 0. This evaluation would be much simpler if we knew the Laurent series about 0 for  $\cot$ . This series can be found by taking the reciprocal of the Taylor series for  $\tan$ , which we obtained in Unit B3, Frame 12. In fact, the first few terms are

$$\cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \dots$$

(4.2) Note that only odd powers of  $z$  appear in this series, since  $\cot$  is an odd function.

For example,

$$\frac{\pi \cot \pi z}{z^2} = \frac{1}{z^3} - \frac{1}{3} \frac{\pi^2}{z} - \dots,$$

so that  $\text{Res}(f, 0) = -\frac{\pi^2}{3}$ , as given in the solution to Example 4.1.

### Problem 4.1

Use the method above, and Equation (4.2), to sum the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

**Remark** Because the Laurent series for  $\cot$  contains odd powers, we can use the method above to sum the series

$$\sum_{n=1}^{\infty} \frac{1}{n^k},$$

whenever  $k$  is *even*. However, when  $k$  is *odd*, the method fails and very little is known about these sums. Even for  $k = 3$ , the exact value of the sum is unknown, although it has been shown (in 1978, by an extremely complicated proof) that it is an irrational number.



In the argument leading to the statement of Theorem 4.1, we saw that the residue at a non-zero integer  $n$  of the function

$$f(z) = \pi \cot \pi z \cdot \phi(z)$$

is given by

$$\operatorname{Res}(f, n) = \phi(n).$$

In fact, this result is also valid for  $n = 0$ , provided that  $\phi$  is analytic at 0. This provides a quick way of evaluating  $\operatorname{Res}(f, 0)$ :

if  $\phi$  is analytic at 0, then

$$\operatorname{Res}(f, 0) = \phi(0).$$

You can use this result in the following problem.

### Problem 4.2

Use the method above to sum the series

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$

(You calculated relevant residues in Problem 1.5(b).)

We conclude this subsection by proving Lemma 4.1.

**Lemma 4.1** For each  $N = 1, 2, \dots$ ,

$$|\cot \pi z| \leq 2, \quad \text{for } z \in S_N,$$

where  $S_N$  is the square contour with vertices at  $(N + \frac{1}{2})(\pm 1 \pm i)$ .

**Proof** First note that if  $z = x + iy$ , then

$$\begin{aligned} |\cot \pi z|^2 &= \frac{|\cos \pi z|^2}{|\sin \pi z|^2} \\ &= \frac{\cos^2 \pi x + \sinh^2 \pi y}{\sin^2 \pi x + \sinh^2 \pi y} \\ &\quad \text{(by Unit A2, Example 4.4(b) and Problem 4.7(b))} \\ &= \frac{1 - \sin^2 \pi x + \sinh^2 \pi y}{\sin^2 \pi x + \sinh^2 \pi y}. \end{aligned} \quad (4.3)$$

Now if  $z = x + iy$  lies on one of the vertical sides of  $S_N$  (see Figure 4.1), then  $x = \pm(N + \frac{1}{2})$ , and so  $\sin \pi x = \pm 1$ . Hence

$$|\cot \pi z|^2 = \frac{\sinh^2 \pi y}{1 + \sinh^2 \pi y} < 1 < 4. \quad (4.4)$$

On the other hand, if  $z = x + iy$  lies on one of the horizontal sides of  $S_N$ , then  $|y| \geq \frac{3}{2}$ , and so

$$|\sinh \pi y| = \sinh \pi |y| \geq \pi |y| \geq 3\pi/2.$$

Now, Identity (4.3) can be written as

$$|\cot \pi z|^2 = 1 + \frac{1 - 2 \sin^2 \pi x}{\sin^2 \pi x + \sinh^2 \pi y};$$

hence, if  $z$  lies on one of the horizontal sides of  $S_N$ , then

$$|\cot \pi z|^2 \leq 1 + \frac{1}{(3\pi/2)^2} < 4. \quad (4.5)$$

The desired result now follows from Inequalities (4.4) and (4.5). ■

$$\begin{aligned} \sinh x &= x + \frac{x^3}{3!} + \dots \\ &\geq x, \quad \text{for } x \geq 0. \end{aligned}$$

## 4.2 Series of the form $\sum_{n=1}^{\infty} (-1)^n \phi(n)$

The method we used above for summing series of the form

$$\sum_{n=1}^{\infty} \phi(n), \quad \text{where } \phi \text{ is an even function,}$$

can easily be adapted to deal with series of the form

$$\sum_{n=1}^{\infty} (-1)^n \phi(n).$$

Examples of such series are

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \dots$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = -\frac{1}{3} + \frac{1}{15} - \frac{1}{35} + \frac{1}{63} - \dots$$

In order to deal with the  $(-1)^n$  term, we replace the function  $g(z) = \pi \cot \pi z$ , whose poles are all simple poles at integer points with residue 1, by the function

$$g(z) = \pi \operatorname{cosec} \pi z = \frac{\pi}{\sin \pi z},$$

whose singularities are also simple poles at the integers  $0, \pm 1, \pm 2, \dots$ , but whose residue at the integer  $n$  is  $(-1)^n$ .

See Example 1.5(b).

To see why this is relevant, consider the function

$$f(z) = \pi \operatorname{cosec} \pi z \cdot \phi(z),$$

where  $\phi$  is an even function analytic on  $\mathbb{C}$  apart from a finite number of poles, none of which occurs at an integer, except (possibly) at 0. Then the residue of  $f$  at a non-zero integer  $n$  is

$$\begin{aligned} \lim_{z \rightarrow n} (z - n)f(z) &= \lim_{z \rightarrow n} ((z - n)\pi \operatorname{cosec} \pi z) \lim_{z \rightarrow n} \phi(z) \\ &= \operatorname{Res}(g, n) \times \phi(n) \\ &= (-1)^n \times \phi(n) = (-1)^n \phi(n). \end{aligned}$$

As before, this result extends to  $n = 0$ ; if  $\phi$  is analytic at 0, then  $\operatorname{Res}(f, 0) = \phi(0)$ .

We now integrate the function  $f(z) = \pi \operatorname{cosec} \pi z \cdot \phi(z)$  around the same square contour  $S_N$ , where  $N$  is an integer large enough for  $S_N$  to contain all the non-zero poles  $\alpha_1, \alpha_2, \dots, \alpha_k$  of  $\phi$  (see Figure 4.1, page 34). It follows from the Residue Theorem, applied to the function  $f(z) = \pi \operatorname{cosec} \pi z \cdot \phi(z)$  and the contour  $S_N$ , that

$$\begin{aligned} \int_{S_N} f(z) dz &= \int_{S_N} \pi \operatorname{cosec} \pi z \cdot \phi(z) dz \\ &= 2\pi i \left( \sum_{n=1}^N \operatorname{Res}(f, n) + \sum_{n=1}^N \operatorname{Res}(f, -n) + \operatorname{Res}(f, 0) + \sum_{j=1}^k \operatorname{Res}(f, \alpha_j) \right) \\ &= 2\pi i \left( \sum_{n=1}^N (-1)^n \phi(n) + \sum_{n=1}^N (-1)^n \phi(-n) + \operatorname{Res}(f, 0) + \sum_{j=1}^k \operatorname{Res}(f, \alpha_j) \right) \\ &= 2\pi i \left( 2 \sum_{n=1}^N (-1)^n \phi(n) + \operatorname{Res}(f, 0) + \sum_{j=1}^k \operatorname{Res}(f, \alpha_j) \right), \end{aligned}$$

since  $\phi$  is an even function.



We now let  $N$  tend to  $\infty$ . If  $\phi$  has been chosen so that the integral on the left tends to 0, that is,

$$\lim_{N \rightarrow \infty} \int_{S_N} \pi \operatorname{cosec} \pi z \cdot \phi(z) dz = 0,$$

then we obtain

$$0 = 2\pi i \left( 2 \sum_{n=1}^{\infty} (-1)^n \phi(n) + \operatorname{Res}(f, 0) + \sum_{j=1}^k \operatorname{Res}(f, \alpha_j) \right).$$

Rearranging this, we obtain

$$\sum_{n=1}^{\infty} (-1)^n \phi(n) = -\frac{1}{2} \left( \operatorname{Res}(f, 0) + \sum_{j=1}^k \operatorname{Res}(f, \alpha_j) \right).$$

Thus we have obtained the sum of the series in terms of the residues of  $f$  at 0 and at the non-zero poles of  $\phi$ .

We summarize this result in the following theorem.

**Theorem 4.2** Let  $\phi$  be an even function which is analytic on  $\mathbb{C}$  except for poles at the points  $\alpha_1, \alpha_2, \dots, \alpha_k$  (none of which is an integer), and possibly at 0, and let  $S_N$  be the square contour with vertices at  $(N + \frac{1}{2})(\pm 1 \pm i)$ . Suppose also that the function  $f(z) = \pi \operatorname{cosec} \pi z \cdot \phi(z)$  is such that

$$\lim_{N \rightarrow \infty} \int_{S_N} f(z) dz = 0. \quad (4.6)$$

Then

$$\sum_{n=1}^{\infty} (-1)^n \phi(n) = -\frac{1}{2} \left( \operatorname{Res}(f, 0) + \sum_{j=1}^k \operatorname{Res}(f, \alpha_j) \right).$$

Condition (4.6) holds for a wide variety of functions  $\phi$ , and is established in particular cases by using the following analogue of Lemma 4.1.

**Lemma 4.2** For each  $N = 1, 2, \dots$ ,

$$|\operatorname{cosec} \pi z| \leq 1, \quad \text{for } z \in S_N,$$

where  $S_N$  is the square contour with vertices at  $(N + \frac{1}{2})(\pm 1 \pm i)$ .

We omit the proof, which is similar to that of Lemma 4.1.

Let us now use Theorem 4.2 to sum a particular series.

### Example 4.2

Sum the series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$ .

### Solution

The function  $\phi(z) = 1/(4z^2 - 1)$  is even and analytic on  $\mathbb{C}$  apart from simple poles at  $\frac{1}{2}, -\frac{1}{2}$ .

The residues of the function

$$f(z) = (\pi \operatorname{cosec} \pi z)/(4z^2 - 1)$$

at  $\frac{1}{2}$  and  $-\frac{1}{2}$  are both  $\pi/4$  (see Problem 1.5(a)).

Since  $\phi$  is analytic at 0,

$$\operatorname{Res}(f, 0) = \phi(0) = -1.$$

We now check Condition (4.6). If  $z$  lies on the contour  $S_N$ , then

$$|z| \geq N + \frac{1}{2},$$

and so, by Lemma 4.2,

$$|f(z)| = \left| \frac{\pi \operatorname{cosec} \pi z}{4z^2 - 1} \right| \leq \frac{\pi}{4(N + \frac{1}{2})^2 - 1}, \quad \text{for } z \in S_N.$$

Hence, by the Estimation Theorem,

$$\left| \int_{S_N} f(z) dz \right| \leq \frac{\pi}{4(N + \frac{1}{2})^2 - 1} \cdot 4(2N + 1),$$

which tends to 0 as  $N \rightarrow \infty$ . Thus Condition (4.6) holds.

It follows from Theorem 4.2 that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} &= -\frac{1}{2} (\operatorname{Res}(f, 0) + \operatorname{Res}(f, \frac{1}{2}) + \operatorname{Res}(f, -\frac{1}{2})) \\ &= -\frac{1}{2} (-1 + \pi/4 + \pi/4) \\ &= \frac{1}{2} - \pi/4. \quad \blacksquare \end{aligned}$$

In the next problem, where the function  $\phi(z) = 1/z^2$  has a pole at 0, it will help you to know that the Laurent series about 0 for cosec is:

$$\operatorname{cosec} z = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \dots \quad (4.7)$$

### Problem 4.3

Use the method above, and Equation (4.7), to sum the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

## EXERCISES

### Section 1

**Exercise 1.1** Find the residues of each of the following functions  $f$  at the given points  $\alpha$ .

$$(a) f(z) = \frac{e^z}{z^7}, \alpha = 0 \quad (b) f(z) = \frac{\cos z}{(z - \pi/2)^2}, \alpha = \pi/2$$

$$(c) f(z) = \frac{e^z}{z^4 - 1}, \alpha = 1, -1, i, -i \quad (d) f(z) = \frac{1}{z^2 - 4}, \alpha = 2, -2$$

$$(e) f(z) = \frac{e^z}{z^3(z^2 - 9)}, \alpha = 0, 3, -3$$

### Section 2

**Exercise 2.1** Use the Residue Theorem to evaluate each of the following contour integrals around the circle  $C = \{z : |z| = 2\}$ .

$$(a) \int_C \frac{\cos z}{(z - \pi/2)^2} dz \quad (b) \int_C \frac{e^z}{z^4 - 1} dz \quad (c) \int_C \frac{e^z}{z^3(z^2 - 9)} dz$$

(Hint: Use results from Exercise 1.1.)



**Exercise 2.2** Use the Residue Theorem to evaluate

$$\int_C \frac{z}{e^z - 1} dz,$$

when

(a)  $C = \{z : |z| = 1\}$ ;      (b)  $C = \{z : |z - 3i| = 4\}$ .

**Exercise 2.3** Evaluate

$$\int_0^{2\pi} \frac{1}{4 \cos^2 t + \sin^2 t} dt.$$

**Exercise 2.4** Prove that if  $a > 1$ , then

$$\int_C \frac{1}{z^2 - 2az + 1} dz = \frac{-\pi i}{\sqrt{a^2 - 1}},$$

where  $C = \{z : |z| = 1\}$ . Hence determine

$$\int_0^{2\pi} \frac{1}{a - \cos t} dt.$$

## Section 3

**Exercise 3.1** Use the methods of the audio tape to evaluate each of the following improper integrals.

(a)  $\int_{-\infty}^{\infty} \frac{t^2}{(t^2 + 4)^2} dt$       (b)  $\int_{-\infty}^{\infty} \frac{\cos t}{t^2 + a^2} dt$ , where  $a > 0$ .

**Exercise 3.2** Use any of the methods available in Section 3 to evaluate each of the following improper integrals.

(a)  $\int_0^{\infty} \frac{1}{(t+1)^2} dt$       (b)  $\int_{-\infty}^{\infty} \frac{t}{t^4 + 1} dt$       (c)  $\int_0^{\infty} \frac{t^2}{(t^2 + 4)^2} dt$   
 (d)  $\int_{-\infty}^{\infty} \frac{t}{t^3 + 1} dt$       (e)  $\int_{-\infty}^{\infty} \frac{t \sin \pi t}{1 - t^2} dt$

## Section 4

**Exercise 4.1** Determine the sum of each of the following series.

(a)  $\sum_{n=1}^{\infty} \frac{1}{4n^2 + 1}$       (b)  $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 + 1}$

(Hint: Use results from Problem 1.5.)

**Exercise 4.2** By taking  $\phi(z) = 1/(z^2 - \alpha^2)$  in Theorem 4.1, obtain the formula

$$\pi \cot \pi \alpha = \frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2 - n^2}, \quad \text{for } \alpha \in \mathbb{C} - \mathbb{Z}.$$

# SOLUTIONS TO THE PROBLEMS

## Section 1

**1.1 (a)** The Laurent series about 0 for the function  $f(z) = (1/z^2) - 3$  is

$$\cdots + \frac{0}{z^3} + \frac{1}{z^2} + \frac{0}{z} - 3 + 0z + 0z^2 + \cdots;$$

since there is no term in  $1/z$ ,  $\text{Res}(f, 0) = 0$ .

**(b)** Since the function  $f(z) = 1/(z-1)$  is analytic at 0, the Laurent series about 0 for  $f$  is a Taylor series and  $\text{Res}(f, 0) = 0$ .

**(c)** The Laurent series about 0 for the function  $f(z) = (\cos z)/z^3$  is

$$\begin{aligned} & \frac{1}{z^3} \left( 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \right) \\ &= \frac{1}{z^3} - \frac{1}{2!z} + \frac{z}{4!} - \frac{z^3}{6!} + \cdots, \end{aligned}$$

and so  $\text{Res}(f, 0) =$  the coefficient of  $z^{-1} = -\frac{1}{2}$ .

**(d)** The Laurent series about 0 for the function  $f(z) = z^2 \sin(1/z)$  is

$$\begin{aligned} & z^2 \left( (1/z) - \frac{(1/z)^3}{3!} + \frac{(1/z)^5}{5!} - \cdots \right) \\ &= z - \frac{1}{3!z} + \frac{1}{5!z^3} - \cdots, \end{aligned}$$

and so  $\text{Res}(f, 0) = -\frac{1}{6}$ .

(Note that  $f$  has an essential singularity, not a pole, at 0.)

**1.2 (a)** Since  $f(z) = 1/(z^2 + 1) = 1/((z-i)(z+i))$ , we let  $z-i = h$ , so that  $z = i+h$ . We obtain, for  $z \neq i, -i$ ,

$$\begin{aligned} \frac{1}{z^2 + 1} &= \frac{1}{(z-i)(z+i)} \\ &= \frac{1}{h(2i+h)} \\ &= \frac{1}{2ih} \left( 1 + \frac{h}{2i} \right)^{-1} \\ &= \frac{1}{2ih} \left( 1 - \left( \frac{h}{2i} \right) + \left( \frac{h}{2i} \right)^2 - \cdots \right) \\ &= \frac{1}{2ih} + \frac{1}{4} - \frac{h}{8i} - \cdots, \quad \text{for } 0 < |h| < 2. \end{aligned}$$

Thus  $\text{Res}(f, 1) =$  the coefficient of  $h^{-1} = \frac{1}{2i} = -\frac{1}{2}i$ .

**(b)** Let  $z - \pi = h$ , so that  $z = \pi + h$ . We obtain, for  $z \neq \pi$ ,

$$\begin{aligned} \frac{ze^{iz}}{(z-\pi)^2} &= \frac{(\pi+h)e^{i(\pi+h)}}{h^2} \\ &= \frac{e^{i\pi}(\pi+h)}{h^2} e^{ih} \\ &= -\left( \frac{\pi+h}{h^2} \right) \left( 1 + ih + \frac{(ih)^2}{2!} + \cdots \right) \\ &\quad \text{(since } e^{i\pi} = -1) \\ &= -\frac{\pi}{h^2} \left( 1 + ih - \frac{h^2}{2!} + \cdots \right) \\ &\quad - \frac{1}{h} \left( 1 + ih - \frac{h^2}{2!} + \cdots \right). \end{aligned}$$

The coefficient of  $h^{-1}$  in this expression is  $-\pi i - 1$ , and so  $\text{Res}(f, \pi) = -\pi i - 1$ .

**1.3 (a)** Since

$$\lim_{z \rightarrow 2i} (z-2i) \cdot \frac{1}{z^2+4} = \lim_{z \rightarrow 2i} \frac{1}{z+2i} = \frac{1}{4i},$$

we deduce, by Theorem 1.1, that  $\text{Res}(f, 2i) = \frac{1}{4i} = \frac{-i}{4}$ .

**(b)** Since

$$\begin{aligned} & \lim_{z \rightarrow 1/3} \left( z - \frac{1}{3} \right) \cdot \frac{1}{z^2(1-z)(1-2z)(1-3z)} \\ &= \lim_{z \rightarrow 1/3} \frac{1}{z^2(1-z)(1-2z)} \cdot \left( -\frac{1}{3} \right) \\ &= \frac{1}{\frac{1}{9} \cdot \frac{2}{3} \cdot \frac{1}{3}} \cdot \left( -\frac{1}{3} \right) = -\frac{27}{2}, \end{aligned}$$

we deduce, by Theorem 1.1, that  $\text{Res}(f, \frac{1}{3}) = -\frac{27}{2}$ .

**1.4 (a)** Let  $g(z) = 1$  and  $h(z) = 2z^2 + 5iz - 2$ .

Then the functions  $g$  and  $h$  are analytic at  $-\frac{1}{2}i$ .

Also,  $h(-\frac{1}{2}i) = 2(-\frac{1}{2}i)^2 + 5i(-\frac{1}{2}i) - 2 = 0$ ,

and  $h'(-\frac{1}{2}i) = 4(-\frac{1}{2}i) + 5i = 3i$ , which is non-zero.

Thus the  $g/h$  Rule applies, and we have

$$\begin{aligned} \text{Res}(f, -\frac{1}{2}i) &= g(-\frac{1}{2}i)/h'(-\frac{1}{2}i) \\ &= 1/(3i) = -\frac{1}{3}i. \end{aligned}$$

**(b)** Let  $g(z) = z + 9$  and  $h(z) = (z^2 + 1)(z^2 + 9)$ .

Then the functions  $g$  and  $h$  are analytic at  $3i$ .

Also,  $h(3i) = ((3i)^2 + 1)((3i)^2 + 9) = -8 \times 0 = 0$ ,

and  $h'(3i) = 2 \cdot 3i \cdot ((3i)^2 + 9) + ((3i)^2 + 1) \cdot 2 \cdot 3i = -48i$ , which is non-zero.

Thus the  $g/h$  Rule applies, and we have

$$\begin{aligned} \text{Res}(f, 3i) &= g(3i)/h'(3i) \\ &= (3i + 9)/(-48i) \\ &= -\frac{1}{16} + \frac{3}{16}i. \end{aligned}$$

Alternatively, we can simplify the calculation of  $h(3i)$  and

$h'(3i)$  by choosing  $g(z) = (z+9)/(z^2+1)$  and

$h(z) = z^2 + 9$ .

Then the functions  $g$  and  $h$  are analytic at  $3i$ .

Also,  $h(3i) = (3i)^2 + 9 = 0$ , and

$h'(3i) = 2 \cdot 3i = 6i$ , which is non-zero.

Thus the  $g/h$  Rule applies, and we have

$$\begin{aligned} \text{Res}(f, 3i) &= g(3i)/h'(3i) \\ &= \frac{3i+9}{(3i)^2+1} / 6i \\ &= (3i+9)/(-48i) \\ &= -\frac{1}{16} + \frac{3}{16}i, \text{ as before.} \end{aligned}$$

**(c)** Let  $g(z) = z^3$  and  $h(z) = z^4 + 1$ .

Then the functions  $g$  and  $h$  are analytic at each given value of  $\alpha$ .

Also, in each case,  $h(\alpha) = \alpha^4 + 1 = (-1) + 1 = 0$ ,

and  $h'(\alpha) = 4\alpha^3$ , which is non-zero.

Thus the  $g/h$  Rule applies, and we have

$$\begin{aligned} \text{Res}(f, \alpha) &= g(\alpha)/h'(\alpha) \\ &= \alpha^3/(4\alpha^3) \\ &= \frac{1}{4}, \text{ for each value of } \alpha. \end{aligned}$$

(Note that, in this example, each singularity has the same residue.)



**1.5 (a)** Let  $g(z) = \pi \operatorname{cosec} \pi z$  and  $h(z) = 4z^2 - 1$ . Then the functions  $g$  and  $h$  are analytic at  $\frac{1}{2}$  and  $-\frac{1}{2}$ . Also,  $h(\frac{1}{2}) = 4(\frac{1}{2})^2 - 1 = 0$ ,

and  $h'(\frac{1}{2}) = 8 \cdot \frac{1}{2} = 4$ , which is non-zero. Thus the  $g/h$  Rule applies, and we have

$$\begin{aligned}\operatorname{Res}(f, \tfrac{1}{2}) &= (\pi \operatorname{cosec} \tfrac{1}{2}\pi) / 4 \\ &= \pi/4.\end{aligned}$$

Similarly,

$$\begin{aligned}\operatorname{Res}(f, -\tfrac{1}{2}) &= (\pi \operatorname{cosec}(-\tfrac{1}{2}\pi)) / (-4) \\ &= \pi/4.\end{aligned}$$

**(b)** Let  $g(z) = \pi \cot \pi z$  and  $h(z) = 4z^2 - 1$ . Then the functions  $g$  and  $h$  are analytic at  $\frac{1}{2}$  and  $-\frac{1}{2}$ , and the calculations of  $h(\frac{1}{2})$  and  $h'(\frac{1}{2})$  are as before. Thus the  $g/h$  Rule applies, and we have

$$\begin{aligned}\operatorname{Res}(f, \tfrac{1}{2}) &= (\pi \cot \tfrac{1}{2}\pi) / 4 \\ &= 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\operatorname{Res}(f, -\tfrac{1}{2}) &= (\pi \cot(-\tfrac{1}{2}\pi)) / (-4) \\ &= 0.\end{aligned}$$

**(c)** Let  $g(z) = \pi \operatorname{cosec} \pi z$  and  $h(z) = 4z^2 + 1$ . Then the functions  $g$  and  $h$  are analytic at  $\frac{1}{2}i$  and  $-\frac{1}{2}i$ . Also,  $h(\frac{1}{2}i) = 4(\frac{1}{2}i)^2 + 1 = 0$ , and  $h'(\frac{1}{2}i) = 8 \cdot \frac{1}{2}i = 4i$ , which is non-zero. Thus the  $g/h$  Rule applies, and we have

$$\begin{aligned}\operatorname{Res}(f, \tfrac{1}{2}i) &= (\pi \operatorname{cosec} \tfrac{1}{2}\pi i) / 4i \\ &= \frac{\pi}{i \sinh \frac{1}{2}\pi} / 4i \\ &= -\frac{\pi}{4 \sinh \frac{1}{2}\pi}.\end{aligned}$$

Similarly,

$$\begin{aligned}\operatorname{Res}(f, -\tfrac{1}{2}i) &= (\pi \operatorname{cosec}(-\tfrac{1}{2}\pi i)) / (-4i) \\ &= -\frac{\pi}{4 \sinh \frac{1}{2}\pi}.\end{aligned}$$

**(d)** Let  $g(z) = \pi \cot \pi z$  and  $h(z) = 4z^2 + 1$ . Then the functions  $g$  and  $h$  are analytic at  $\frac{1}{2}i$  and  $-\frac{1}{2}i$ , and the calculations of  $h(\frac{1}{2}i)$  and  $h'(\frac{1}{2}i)$  are as before. Thus the  $g/h$  Rule applies, and we have

$$\begin{aligned}\operatorname{Res}(f, \tfrac{1}{2}i) &= (\pi \cot \tfrac{1}{2}\pi i) / 4i \\ &= \frac{\pi \cosh \frac{1}{2}\pi}{i \sinh \frac{1}{2}\pi} / 4i \\ &= -\frac{\pi \cosh \frac{1}{2}\pi}{4 \sinh \frac{1}{2}\pi}.\end{aligned}$$

Similarly,

$$\begin{aligned}\operatorname{Res}(f, -\tfrac{1}{2}i) &= (\pi \cot(-\tfrac{1}{2}\pi i)) / (-4i) \\ &= -\frac{\pi \cosh \frac{1}{2}\pi}{4 \sinh \frac{1}{2}\pi}.\end{aligned}$$

**1.6 (a)** The function

$$f(z) = \frac{z+2}{z^3(z+4)}$$

has a simple pole at the point  $-4$ . If we cover up the term  $(z+4)$  and evaluate what remains at  $-4$ , we obtain

$$\operatorname{Res}(f, -4) = \frac{-4+2}{(-4)^3} = \frac{1}{32}.$$

**(b)** The function  $f(z) = (\cos z)/(ze^z)$  has a simple pole at the point 0. Applying the Cover-up Rule, we obtain

$$\operatorname{Res}(f, 0) = \frac{\cos 0}{e^0} = 1.$$

**(c)** We can write

$$f(z) = \frac{-1}{3z^2(1-z)(1-2z)(z-\frac{1}{3})}.$$

The function  $f$  has a simple pole at the point  $\frac{1}{3}$ . Applying the Cover-up Rule, we obtain

$$\operatorname{Res}(f, \tfrac{1}{3}) = \frac{-1}{3 \cdot \frac{1}{9} \cdot \frac{2}{3} \cdot \frac{1}{3}} = -\frac{27}{2}.$$

This agrees with the answer to Problem 1.3(b).

**(d)** We can write

$$f(z) = \frac{\sin z}{z \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)^3}.$$

The function  $f$  has a removable singularity at 0. Applying the Cover-up Rule, we obtain

$$\operatorname{Res}(f, 0) = \frac{\sin 0}{1^3} = 0 \quad (\text{as expected}).$$

$$\begin{aligned}\mathbf{1.7 (a)} \quad f(z) &= \frac{z+2}{z^3(z+4)} \\ &= \frac{1}{4z^3}(z+2)\left(1+\frac{z}{4}\right)^{-1} \\ &= \frac{1}{4z^3}(z+2)\left(1-\left(\frac{z}{4}\right)+\left(\frac{z}{4}\right)^2-\left(\frac{z}{4}\right)^3+\cdots\right), \\ &\quad \text{for } 0 < |z| < 4, \\ &= \frac{1}{4z^3}\left(\left(z-\frac{z^2}{4}+\frac{z^3}{16}-\cdots\right)\right. \\ &\quad \left.+\left(2-\frac{z}{2}+\frac{z^2}{8}-\frac{z^3}{32}+\cdots\right)\right),\end{aligned}$$

and so

$$\operatorname{Res}(f, 0) = \frac{1}{4}\left(-\frac{1}{4}+\frac{1}{8}\right) = -\frac{1}{32}.$$

**(b)** (In this case, for variety, we choose not to use the substitution  $z-1=h$  explicitly.) We express  $f(z)$  in terms of  $z-1$ . Now, for  $z \neq 1$ ,

$$\begin{aligned}f(z) &= \frac{1+e^{2z}}{(z-1)^4} \\ &= \frac{1+e^{2(z-1)}}{(z-1)^4} \\ &= \frac{1}{(z-1)^4}\left(1+e^2\left(1+2(z-1)+\frac{4(z-1)^2}{2!}\right.\right. \\ &\quad \left.\left.+\frac{8(z-1)^3}{3!}+\cdots\right)\right),\end{aligned}$$

$$\text{and so } \operatorname{Res}(f, 1) = \frac{8e^2}{3!} = \frac{4}{3}e^2.$$

**1.8 (a)** Since the function  $f$  has a pole of order two at the point  $\pi$ , we apply Theorem 1.2 with  $k=2$ . We obtain

$$\begin{aligned}\operatorname{Res}(f, \pi) &= \lim_{z \rightarrow \pi} \left(\frac{d}{dz}(ze^{iz})\right) \\ &= \lim_{z \rightarrow \pi} (e^{iz} + iz e^{iz}) \\ &= e^{i\pi} + i\pi e^{i\pi} = -1 - i\pi.\end{aligned}$$

(Note that this answer agrees with that of Problem 1.2(b).)

(b) Since the function  $f$  has a pole of order three at the point 0, we apply Theorem 1.2 with  $k = 3$ . We obtain

$$\begin{aligned}\operatorname{Res}(f, 0) &= \frac{1}{2!} \lim_{z \rightarrow 0} \left( \frac{d^2}{dz^2} \left( \frac{z+2}{z+4} \right) \right) \\ &= \frac{1}{2} \lim_{z \rightarrow 0} \left( \frac{d}{dz} \left( \frac{2}{(z+4)^2} \right) \right) \\ &= \frac{1}{2} \lim_{z \rightarrow 0} \left( \frac{-4}{(z+4)^3} \right) \\ &= \frac{1}{2} \left( \frac{-4}{4^3} \right) = -\frac{1}{32}.\end{aligned}$$

(Note that this answer agrees with that of Problem 1.7(a).)

**1.9** If  $f$  has a pole of order  $k$  at  $\alpha$ , then its Laurent series about  $\alpha$  is

$$f(z) = \frac{a_{-k}}{(z-\alpha)^k} + \cdots + \frac{a_{-1}}{z-\alpha} + a_0 + a_1(z-\alpha) + \cdots,$$

where  $a_{-k} \neq 0$ , and so

$$(z-\alpha)^k f(z) = a_{-k} + \cdots + a_{-1}(z-\alpha)^{k-1} + a_0(z-\alpha)^k + a_1(z-\alpha)^{k+1} + \cdots.$$

Differentiating this  $k-1$  times, we obtain

$$\begin{aligned}\frac{d^{k-1}}{dz^{k-1}} ((z-\alpha)^k f(z)) &= (k-1)! a_{-1} + k! a_0 (z-\alpha) \\ &\quad + \frac{(k+1)!}{2!} a_1 (z-\alpha)^2 + \cdots.\end{aligned}$$

Dividing by  $(k-1)!$  and taking the limit as  $z$  tends to  $\alpha$  gives

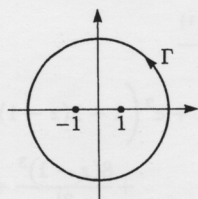
$$\frac{1}{(k-1)!} \lim_{z \rightarrow \alpha} \left( \frac{d^{k-1}}{dz^{k-1}} ((z-\alpha)^k f(z)) \right) = a_{-1},$$

as required.

## Section 2

**2.1** Let  $f(z) = \frac{\sin z}{z^2 - 1}$ .

(a) The function  $f(z) = (\sin z)/(z^2 - 1)$  is analytic on  $\mathbb{C}$  apart from simple poles at 1 and  $-1$ , both of which lie inside the circle  $\Gamma = \{z : |z| = 3\}$ , as shown in the figure.



Using the  $g/h$  Rule, with  $g(z) = \sin z$ ,  $h(z) = z^2 - 1$  and  $h'(z) = 2z$ , we obtain

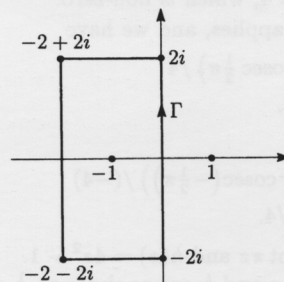
$$\operatorname{Res}(f, 1) = (\sin 1)/2 = \frac{1}{2} \sin 1;$$

$$\operatorname{Res}(f, -1) = (\sin(-1))/(-2) = \frac{1}{2} \sin 1.$$

Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ ,

$$\begin{aligned}\int_{\Gamma} \frac{\sin z}{z^2 - 1} dz &= 2\pi i (\operatorname{Res}(f, 1) + \operatorname{Res}(f, -1)) \\ &= 2\pi i \left( \frac{1}{2} \sin 1 + \frac{1}{2} \sin 1 \right) \\ &= 2\pi i \sin 1.\end{aligned}$$

(b) In this case, the simple pole at  $-1$  lies inside the rectangular contour  $\Gamma$  with vertices at  $-2i$ ,  $2i$ ,  $-2 + 2i$ ,  $-2 - 2i$ ; that at 1 lies outside  $\Gamma$  (see the figure).



It follows from the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ , as above, that

$$\begin{aligned}\int_{\Gamma} \frac{\sin z}{z^2 - 1} dz &= 2\pi i \operatorname{Res}(f, -1) \\ &= 2\pi i \cdot \frac{1}{2} \sin 1 = \pi i \sin 1.\end{aligned}$$

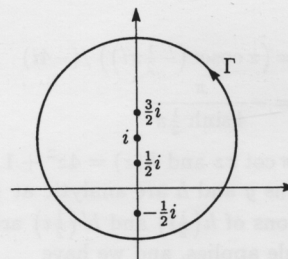
**2.2** The function  $f(z) = (z+2)/(4z^2 + k^2)$  is analytic on  $\mathbb{C}$  apart from simple poles at  $\frac{1}{2}ki$  and  $-\frac{1}{2}ki$ .

Using the  $g/h$  Rule with  $g(z) = z+2$ ,  $h(z) = 4z^2 + k^2$  and  $h'(z) = 8z$ , we obtain

$$\begin{aligned}\operatorname{Res}\left(f, \frac{1}{2}ki\right) &= \left(\frac{1}{2}ki + 2\right)/8\left(\frac{1}{2}ki\right) \\ &= (ki + 4)/8ki;\end{aligned}$$

$$\begin{aligned}\operatorname{Res}\left(f, -\frac{1}{2}ki\right) &= \left(-\frac{1}{2}ki + 2\right)/8\left(-\frac{1}{2}ki\right) \\ &= (ki - 4)/8ki.\end{aligned}$$

The figure shows the circle  $\Gamma = \{z : |z - i| = 2\}$ .



(a) If  $k = 1$ , then both poles lie inside  $\Gamma$ . Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ ,

$$\begin{aligned}I &= 2\pi i (\operatorname{Res}(f, \frac{1}{2}i) + \operatorname{Res}(f, -\frac{1}{2}i)) \\ &= 2\pi i \left( \frac{i+4}{8i} + \frac{i-4}{8i} \right) \\ &= \frac{\pi i}{2}.\end{aligned}$$

(b) If  $k = 3$ , then the pole at  $\frac{3}{2}i$  lies inside  $\Gamma$  and that at  $-\frac{3}{2}i$  lies outside. Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ ,

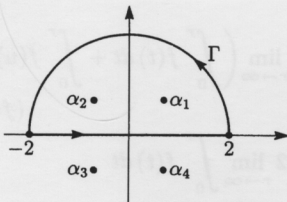
$$\begin{aligned}I &= 2\pi i \operatorname{Res}\left(f, \frac{3}{2}i\right) \\ &= 2\pi i \left( \frac{3i+4}{24i} \right) \\ &= \frac{\pi}{3} + \frac{\pi i}{4}.\end{aligned}$$

(c) If  $k = 7$ , then neither pole lies inside  $\Gamma$ . Hence, by Cauchy's Theorem (with  $\mathcal{R} = \{z : -7/2 < \operatorname{Im} z < 7/2\}$ ), the value of the integral is 0.



**2.3** The function  $f(z) = z^3/(z^4 + 1)$  is analytic on  $\mathbb{C}$  apart from simple poles at the points where the denominator is zero, that is, at  $\alpha_1 = e^{\pi i/4}$ ,  $\alpha_2 = e^{3\pi i/4}$ ,  $\alpha_3 = e^{5\pi i/4}$  and  $\alpha_4 = e^{7\pi i/4}$ .

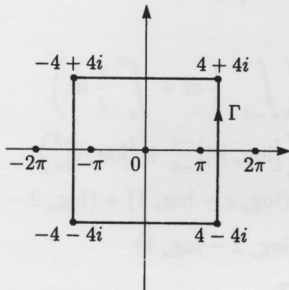
Of these poles, only the first two lie inside the given semicircular contour  $\Gamma$  (see the figure), and the residue at each of these poles is  $\frac{1}{4}$  (see Problem 1.4(c)).



Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ ,

$$\begin{aligned} \int_{\Gamma} \frac{z^3}{z^4 + 1} dz &= 2\pi i (\text{Res}(f, e^{\pi i/4}) + \text{Res}(f, e^{3\pi i/4})) \\ &= 2\pi i \left(\frac{1}{4} + \frac{1}{4}\right) = \pi i. \end{aligned}$$

**2.4** The function  $f(z) = (1+z)/\sin z$  is analytic on  $\mathbb{C}$  apart from simple poles at the points where  $\sin z = 0$ , that is, at  $z = n\pi$ , where  $n$  is an integer. (So  $f$  has infinitely many simple poles.) Of these poles, only those at  $0, \pi$  and  $-\pi$  lie inside the given square contour  $\Gamma$  (see the figure). None of the poles lies on  $\Gamma$ .



Also, by the  $g/h$  Rule, with  $g(z) = 1 + z$ ,  $h(z) = \sin z$  and  $h'(z) = \cos z$ ,

$$\text{Res}(f, 0) = (1 + 0)/\cos 0 = 1,$$

$$\text{Res}(f, \pi) = (1 + \pi)/\cos \pi = -1 - \pi,$$

and

$$\text{Res}(f, -\pi) = (1 - \pi)/\cos(-\pi) = -1 + \pi.$$

Hence, by the Residue Theorem with  $\mathcal{R} = \{z : |z| < 6\}$ , for example,

$$\begin{aligned} \int_{\Gamma} \frac{1+z}{\sin z} dz &= 2\pi i (\text{Res}(f, 0) + \text{Res}(f, \pi) + \text{Res}(f, -\pi)) \\ &= 2\pi i (1 + (-1 - \pi) + (-1 + \pi)) \\ &= -2\pi i. \end{aligned}$$

(Note that the choice of  $\mathcal{R} = \mathbb{C}$  is not acceptable since  $f$  has infinitely many poles in  $\mathbb{C}$ .)

**2.5** Using the given strategy, we obtain

$$\begin{aligned} &\int_0^{2\pi} \frac{1}{25 \cos^2 t + 9 \sin^2 t} dt \\ &= \int_C \frac{1}{25 \cdot \frac{1}{4}(z + z^{-1})^2 + 9 \cdot (-\frac{1}{4})(z - z^{-1})^2} \cdot \frac{1}{iz} dz \\ &= \int_C \frac{1}{iz \left( \frac{25}{4}(z^2 + 2 + z^{-2}) - \frac{9}{4}(z^2 - 2 + z^{-2}) \right)} dz \\ &= \int_C \frac{1}{iz(4z^2 + 17 + 4z^{-2})} dz \\ &= \int_C \frac{z}{i(4z^4 + 17z^2 + 4)} dz \\ &= \frac{1}{4i} \int_C \frac{z}{(z^2 + 4)(z^2 + \frac{1}{4})} dz, \end{aligned}$$

as required.

Now, the singularities of the function  $f(z) = z/((z^2 + 4)(z^2 + \frac{1}{4}))$  are simple poles at  $2i, -2i, \frac{1}{2}i$  and  $-\frac{1}{2}i$ . Of these poles, only the last two lie inside the unit circle  $C$ ; the other two lie outside  $C$ .

Using the  $g/h$  Rule, with  $g(z) = z/(z^2 + 4)$ ,  $h(z) = z^2 + \frac{1}{4}$  and  $h'(z) = 2z$ , we obtain

$$\text{Res}(f, \frac{1}{2}i) = \frac{\frac{1}{2}i}{(\frac{1}{2}i)^2 + 4} / 2 \cdot (\frac{1}{2}i) = \frac{2}{15};$$

$$\text{Res}(f, -\frac{1}{2}i) = \frac{-\frac{1}{2}i}{(-\frac{1}{2}i)^2 + 4} / 2 \cdot (-\frac{1}{2}i) = \frac{2}{15}.$$

Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ , the value of the original integral is

$$\begin{aligned} &2\pi i \cdot \frac{1}{4i} (\text{Res}(f, \frac{1}{2}i) + \text{Res}(f, -\frac{1}{2}i)) \\ &= \frac{\pi}{2} \left( \frac{2}{15} + \frac{2}{15} \right) = \frac{2\pi}{15}. \end{aligned}$$

**2.6 (a)** The residue of the function  $f(z) = (z^2 + 1)^n / z^{n+1}$  at the point 0 is the coefficient of  $z^{-1}$  in the expansion of  $(z^2 + 1)^n / z^{n+1}$ ; this is just the coefficient of  $z^n$  in the expansion of  $(z^2 + 1)^n$ , which is

$$\binom{n}{\frac{1}{2}n} \text{ if } n \text{ is even, and } 0 \text{ if } n \text{ is odd.}$$

(b) Using the strategy, we obtain

$$\begin{aligned} \int_0^{2\pi} \cos^n t dt &= \int_C \left( \frac{1}{2}(z + z^{-1}) \right)^n \frac{1}{iz} dz \\ &= \frac{1}{2^n i} \int_C \frac{(z^2 + 1)^n}{z^{n+1}} dz. \end{aligned}$$

The only singularity of the function  $f(z) = (z^2 + 1)^n / z^{n+1}$  is a pole of order  $n + 1$  at the point 0 (which lies inside the unit circle  $C$ ), and the residue of  $f$  at 0 is as given in part (a). Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ , the value of the original integral is

$$2\pi i \cdot \frac{1}{2^n i} \cdot \text{Res}(f, 0) = \begin{cases} \frac{\pi}{2^{n-1}} \binom{n}{\frac{1}{2}n}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

## Section 3

**3.1** First note that the function  $f(z) = 1/\sqrt{z}$  is defined on  $]0, \infty[$ . Let  $\varepsilon$  be any positive number. Then

$$\begin{aligned} \left| \frac{1}{\sqrt{r}} - 0 \right| < \varepsilon &\iff \frac{1}{\sqrt{r}} < \varepsilon \\ &\iff \frac{1}{r} < \varepsilon^2 \\ &\iff r > 1/\varepsilon^2. \end{aligned}$$

Let  $N = [1/\varepsilon^2]$ . Then

$$\left| \frac{1}{\sqrt{r}} - 0 \right| < \varepsilon, \quad \text{for all } r > N.$$

Thus  $\lim_{r \rightarrow \infty} \frac{1}{\sqrt{r}} = 0$ .

**Erratum:**  
You need to take  $N = [1/\varepsilon^2] + 1$ , or any larger integer. (The choice  $N = [1/\varepsilon^2]$  would be satisfactory when proving that the sequence  $\{1/\sqrt{n}\}$  is null, because  $n$  takes only positive integer values.)

**3.2** Let

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

where  $a_n \neq 0$ , and

$$q(z) = b_0 + b_1 z + \cdots + b_m z^m,$$

where  $b_m \neq 0$ . Then

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{p(r)}{q(r)} &= \lim_{r \rightarrow \infty} \frac{p(r)/r^m}{q(r)/r^m} \\ &= \lim_{r \rightarrow \infty} \frac{a_0 r^{-m} + a_1 r^{-m+1} + \cdots + a_n r^{-m+n}}{b_0 r^{-m} + b_1 r^{-m+1} + \cdots + b_{m-1} r^{-1} + b_m}. \end{aligned}$$

Hence, if the degree of  $q$  exceeds the degree of  $p$  (that is, if  $m > n$ ) then, since  $\lim_{r \rightarrow \infty} 1/r = 0$ , it follows from the Combination Rules that

$$\lim_{r \rightarrow \infty} \frac{p(r)}{q(r)} = \frac{0 + 0 + \cdots + 0}{0 + 0 + \cdots + 0 + b_m} = 0.$$

**3.3 (a)** 
$$\int_{-\infty}^{\infty} \sin t \, dt = \lim_{r \rightarrow \infty} \int_{-r}^r \sin t \, dt$$
$$= \lim_{r \rightarrow \infty} [-\cos t]_{-r}^r = \lim_{r \rightarrow \infty} 0 = 0.$$

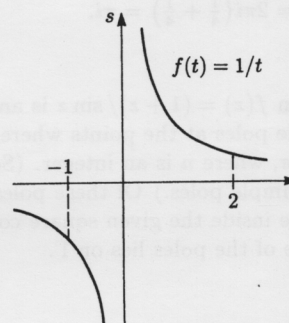
**(b)** 
$$\int_1^{\infty} \frac{1}{t^p} \, dt = \lim_{r \rightarrow \infty} \int_1^r \frac{1}{t^p} \, dt$$
$$= \lim_{r \rightarrow \infty} \left[ \frac{1}{1-p} \cdot \frac{1}{t^{p-1}} \right]_1^r, \quad \text{since } p > 1,$$
$$= \lim_{r \rightarrow \infty} \frac{1}{1-p} \left( \frac{1}{r^{p-1}} - 1 \right)$$
$$= 1/(p-1).$$

**(c)** 
$$\int_0^{\infty} e^{-t} \, dt = \lim_{r \rightarrow \infty} \int_0^r e^{-t} \, dt$$
$$= \lim_{r \rightarrow \infty} [-e^{-t}]_0^r$$
$$= \lim_{r \rightarrow \infty} (-e^{-r} - (-1)) = 1.$$

**3.4** If  $f$  is an even function, then

$$\begin{aligned} \int_{-\infty}^{\infty} f(t) \, dt &= \lim_{r \rightarrow \infty} \int_{-r}^r f(t) \, dt \\ &= \lim_{r \rightarrow \infty} \left( \int_0^r f(t) \, dt + \int_{-r}^0 f(t) \, dt \right) \\ &= \lim_{r \rightarrow \infty} \left( \int_0^r f(t) \, dt + \int_r^0 f(-u)(-du) \right) \\ &\quad (t = -u) \\ &= \lim_{r \rightarrow \infty} \left( \int_0^r f(t) \, dt + \int_0^r f(u) \, du \right) \\ &\quad (f(-u) = f(u)) \\ &= 2 \lim_{r \rightarrow \infty} \int_0^r f(t) \, dt \\ &= 2 \int_0^{\infty} f(t) \, dt. \end{aligned}$$

**3.5** The function  $f(t) = 1/t$  is continuous at all points of  $[-1, 2]$  except at the point 0, as shown in the figure.



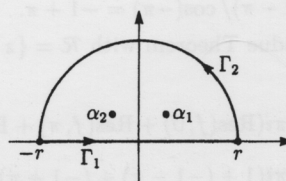
Thus

$$\begin{aligned} \int_{-1}^2 \frac{1}{t} \, dt &= \lim_{\varepsilon \rightarrow 0} \left( \int_{-1}^{-\varepsilon} \frac{1}{t} \, dt + \int_{\varepsilon}^2 \frac{1}{t} \, dt \right) \\ &= \lim_{\varepsilon \rightarrow 0} \left( [\log_e |t|]_{-1}^{-\varepsilon} + [\log_e t]_{\varepsilon}^2 \right) \\ &= \lim_{\varepsilon \rightarrow 0} ((\log_e \varepsilon - \log_e 1) + (\log_e 2 - \log_e \varepsilon)) \\ &= \lim_{\varepsilon \rightarrow 0} (\log_e 2 - \log_e 1) \\ &= \log_e 2. \end{aligned}$$

**3.6 (a)** Consider the contour integral

$$I = \int_{\Gamma} \frac{1}{z^4 + 1} \, dz,$$

where  $\Gamma = \Gamma_1 + \Gamma_2$  is the contour shown in the figure.



The function  $f(z) = 1/(z^4 + 1)$  is analytic on  $\mathbb{C}$  apart from simple poles at

$$\alpha_1 = e^{\pi i/4}, \quad \alpha_2 = e^{3\pi i/4}, \quad \alpha_3 = e^{5\pi i/4}, \quad \alpha_4 = e^{7\pi i/4}.$$

The poles at  $\alpha_1 = e^{\pi i/4}$  and  $\alpha_2 = e^{3\pi i/4}$  lie inside  $\Gamma$ , for  $r > 1$ ; the other two lie outside  $\Gamma$ .



(b) By the Residue Theorem,

$$\begin{aligned} I &= 2\pi i (\text{Res}(f, e^{\pi i/4}) + \text{Res}(f, e^{3\pi i/4})) \\ &= 2\pi i \left( \frac{1}{4} e^{-3\pi i/4} + \frac{1}{4} e^{-9\pi i/4} \right) \quad (\text{by the } g/h \text{ Rule}) \\ &= \frac{\pi i}{2} \left( \left( \cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} \right) + \left( \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right) \right) \\ &= \pi/\sqrt{2}. \end{aligned}$$

(c) Splitting up the integral gives

$$\begin{aligned} I &= \int_{\Gamma_1} \frac{1}{z^4 + 1} dz + \int_{\Gamma_2} \frac{1}{z^4 + 1} dz \\ &= \int_{-r}^r \frac{1}{t^4 + 1} dt + \int_{\Gamma_2} \frac{1}{z^4 + 1} dz. \end{aligned} \quad (1)$$

(d) By the Estimation Theorem and the Triangle Inequality,

$$\begin{aligned} \left| \int_{\Gamma_2} \frac{1}{z^4 + 1} dz \right| &\leq \frac{1}{r^4 - 1} \times \pi r \\ &= \frac{\pi r}{r^4 - 1}, \quad \text{for } r > 1. \end{aligned}$$

(The length of  $\Gamma_2$  is  $\pi r$  and, by the Triangle Inequality,  $|z^4 + 1| \geq r^4 - 1$  on  $\Gamma_2$ .)

(e) We now let  $r \rightarrow \infty$  in Equation (1):

$$\begin{aligned} \lim_{r \rightarrow \infty} \int_{-r}^r \frac{1}{t^4 + 1} dt + \lim_{r \rightarrow \infty} \int_{\Gamma_2} \frac{1}{z^4 + 1} dz &= \pi/\sqrt{2} \\ &\quad (\text{by part (b)}). \end{aligned}$$

From part (d),

$$\lim_{r \rightarrow \infty} \int_{\Gamma_2} \frac{1}{z^4 + 1} dz = 0;$$

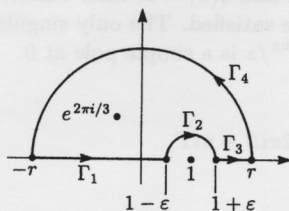
hence

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{t^4 + 1} dt &= \lim_{r \rightarrow \infty} \int_{-r}^r \frac{1}{t^4 + 1} dt \\ &= \pi/\sqrt{2}. \end{aligned}$$

**3.7 (a)** Consider the contour integral

$$I = \int_{\Gamma} \frac{z}{z^3 - 1} dz,$$

where  $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$  is the contour shown in the figure.



The function  $f(z) = z/(z^3 - 1)$  is analytic on  $\mathbb{C}$  apart from simple poles at  $1$ ,  $e^{2\pi i/3}$  and  $e^{4\pi i/3}$ . The pole at  $e^{2\pi i/3}$  lies inside  $\Gamma$ , for  $r > 1$ ; the other two lie outside  $\Gamma$ .

(b) By the Residue Theorem,

$$\begin{aligned} I &= 2\pi i \times (\text{residue at } e^{2\pi i/3}) \\ &= 2\pi i \times \frac{e^{2\pi i/3}}{3e^{4\pi i/3}} \quad (\text{by the } g/h \text{ Rule}) \\ &= \frac{2\pi i}{3} e^{-2\pi i/3} \\ &= \frac{\pi i}{3} (-1 - i\sqrt{3}). \end{aligned}$$

(c) Splitting up the integral gives

$$\begin{aligned} I &= \int_{-r}^{1-\epsilon} \frac{t}{t^3 - 1} dt + \int_{1+\epsilon}^r \frac{t}{t^3 - 1} dt \\ &\quad + \int_{\Gamma_2} \frac{z}{z^3 - 1} dz + \int_{\Gamma_4} \frac{z}{z^3 - 1} dz. \end{aligned} \quad (2)$$

(d) By the Estimation Theorem and the Triangle Inequality,

$$\begin{aligned} \left| \int_{\Gamma_4} \frac{z}{z^3 - 1} dz \right| &\leq \frac{r}{r^3 - 1} \times \pi r \\ &= \frac{\pi r^2}{r^3 - 1}, \quad \text{for } r > 1. \end{aligned}$$

(e) We now let  $\epsilon \rightarrow 0$  and  $r \rightarrow \infty$  in Equation (2). We have

$$\int_{-r}^{1-\epsilon} \frac{t}{t^3 - 1} dt + \int_{1+\epsilon}^r \frac{t}{t^3 - 1} dt \rightarrow \int_{-\infty}^{\infty} \frac{t}{t^3 - 1} dt,$$

provided that this limit exists. The existence follows because

$$\int_{\Gamma_4} \frac{z}{z^3 - 1} dz \rightarrow 0 \text{ as } r \rightarrow \infty \quad (\text{by part (d)}),$$

and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_2} \frac{z}{z^3 - 1} dz &= -\pi i \times (\text{residue at } 1) \\ &\quad (\text{by the Round-the-Pole Lemma}) \\ &= -\pi i \times \frac{1}{3} \quad (\text{by the } g/h \text{ Rule}) \\ &= -\frac{1}{3}\pi i. \end{aligned}$$

Equation (2) becomes

$$\int_{-\infty}^{\infty} \frac{t}{t^3 - 1} dt - \frac{1}{3}\pi i + 0 = \frac{\pi i}{3} (-1 - i\sqrt{3}) \quad (\text{by part (b)}).$$

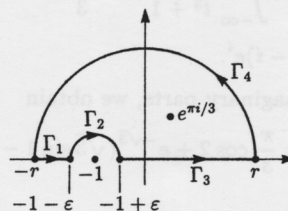
Hence

$$\int_{-\infty}^{\infty} \frac{t}{t^3 - 1} dt = \frac{\pi\sqrt{3}}{3}.$$

**3.8 (a)** Consider the contour integral

$$I = \int_{\Gamma} \frac{e^{2iz}}{z^3 + 1} dz,$$

where  $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$  is the contour shown in the figure.



The function  $f(z) = e^{2iz}/(z^3 + 1)$  is analytic on  $\mathbb{C}$  apart from simple poles at  $-1$ ,  $e^{\pi i/3}$  and  $e^{-\pi i/3}$ . The pole at  $e^{\pi i/3}$  lies inside  $\Gamma$ , for  $r > 1$ ; the other two lie outside  $\Gamma$ .

(b) By the Residue Theorem,

$$\begin{aligned} I &= 2\pi i \times (\text{residue at } e^{\pi i/3}) \\ &= 2\pi i \times \frac{\exp(2ie^{\pi i/3})}{3e^{2\pi i/3}} \quad (\text{by the } g/h \text{ Rule}) \\ &= \frac{2\pi i \exp(-\sqrt{3} + i)}{3(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)} = \frac{\pi}{3} e^{-\sqrt{3}} (\sqrt{3} - i)e^i. \end{aligned}$$

(c) Splitting up the integrals gives

$$\begin{aligned}
 I &= \int_{-r}^{-1-\varepsilon} \frac{e^{2it}}{t^3+1} dt + \int_{-1+\varepsilon}^r \frac{e^{2it}}{t^3+1} dt \\
 &\quad + \int_{\Gamma_2} \frac{e^{2iz}}{z^3+1} dz + \int_{\Gamma_4} \frac{e^{2iz}}{z^3+1} dz \\
 &= \int_{-r}^{-1-\varepsilon} \frac{\cos 2t}{t^3+1} dt + \int_{-1+\varepsilon}^r \frac{\cos 2t}{t^3+1} dt \\
 &\quad + i \int_{-r}^{-1-\varepsilon} \frac{\sin 2t}{t^3+1} dt + i \int_{-1+\varepsilon}^r \frac{\sin 2t}{t^3+1} dt \\
 &\quad + \int_{\Gamma_2} \frac{e^{2iz}}{z^3+1} dz + \int_{\Gamma_4} \frac{e^{2iz}}{z^3+1} dz, \quad (3)
 \end{aligned}$$

since  $e^{2it} = \cos 2t + i \sin 2t$ .

(d) By the Estimation Theorem and the Triangle Inequality, and the fact that

$$|e^{2iz}| = e^{-2y} \leq 1, \quad \text{for } y \geq 0,$$

$$\begin{aligned}
 \left| \int_{\Gamma_4} \frac{e^{2iz}}{z^3+1} dz \right| &\leq \frac{1}{r^3-1} \times \pi r \\
 &= \frac{\pi r}{r^3-1}, \quad \text{for } r > 1.
 \end{aligned}$$

(e) We now let  $\varepsilon \rightarrow 0$  and  $r \rightarrow \infty$  in Equation (3).

We have

$$\begin{aligned}
 \int_{-r}^{-1-\varepsilon} \frac{\cos 2t}{t^3+1} dt + \int_{-1+\varepsilon}^r \frac{\cos 2t}{t^3+1} dt &\rightarrow \int_{-\infty}^{\infty} \frac{\cos 2t}{t^3+1} dt, \\
 \int_{-r}^{-1-\varepsilon} \frac{\sin 2t}{t^3+1} dt + \int_{-1+\varepsilon}^r \frac{\sin 2t}{t^3+1} dt &\rightarrow \int_{-\infty}^{\infty} \frac{\sin 2t}{t^3+1} dt,
 \end{aligned}$$

provided that these limits exist. Their existence follows because

$$\int_{\Gamma_4} \frac{e^{2iz}}{z^3+1} dz \rightarrow 0 \quad (\text{by part (d)}),$$

and

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_2} \frac{e^{2iz}}{z^3+1} dz &= -\pi i \times (\text{residue at } -1) \\
 &\quad (\text{by the Round-the-Pole Lemma}) \\
 &= -\pi i \times \frac{e^{-2i}}{3 \times (-1)^2} \quad (\text{by the } g/h \text{ Rule}) \\
 &= -\frac{\pi}{3} \sin 2 - i \frac{\pi}{3} \cos 2.
 \end{aligned}$$

Equation (3) becomes

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{\cos 2t}{t^3+1} dt + i \int_{-\infty}^{\infty} \frac{\sin 2t}{t^3+1} dt &- \frac{\pi}{3} \sin 2 - i \frac{\pi}{3} \cos 2 = 0 \\
 &= \frac{\pi}{3} e^{-\sqrt{3}} (\sqrt{3} - i) e^i.
 \end{aligned}$$

Hence, equating imaginary parts, we obtain

$$\int_{-\infty}^{\infty} \frac{\sin 2t}{t^3+1} dt = \frac{\pi}{3} (\cos 2 + e^{-\sqrt{3}} (\sqrt{3} \sin 1 - \cos 1)).$$

**3.9** Let  $p(z) = 1$  and  $q(z) = z(z-1)(z^2+1)$ ; then Conditions 1 and 2 of Theorem 3.3 are satisfied.

The only singularities of the function  $p/q$  are simple poles at 0, 1,  $i$  and  $-i$ . Of these poles,  $i$  lies in the upper-half plane, and 0 and 1 lie on the real axis.

By Theorem 3.3,

$$\int_{-\infty}^{\infty} \frac{1}{t(t-1)(t^2+1)} dt = 2\pi i S + \pi i T,$$

where

$$\begin{aligned}
 S &= \text{Res}(p/q, i) \\
 &= \frac{1}{i(i-1)2i} \quad (\text{by the } g/h \text{ Rule with } h(z) = z^2+1) \\
 &= \frac{1}{2(1-i)}
 \end{aligned}$$

and

$$\begin{aligned}
 T &= \text{Res}(p/q, 0) + \text{Res}(p/q, 1) \\
 &= \frac{1}{(-1) \cdot 1} + \frac{1}{1 \cdot 2} = -\frac{1}{2} \quad (\text{by the Cover-up Rule}).
 \end{aligned}$$

Thus

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{1}{t(t-1)(t^2+1)} dt &= 2\pi i \cdot \frac{1}{2(1-i)} + \pi i \left(-\frac{1}{2}\right) \\
 &= -\pi/2.
 \end{aligned}$$

**3.10** Let  $p(z) = 1$  and  $q(z) = (z^2+a^2)(z^2+b^2)$ , where  $a, b > 0$  and  $a \neq b$ ; then Conditions 1 and 2 of Theorem 3.3 are satisfied.

The only singularities of the function  $p/q$  are simple poles at  $ai$ ,  $-ai$ ,  $bi$  and  $-bi$ . Of these poles, only  $ai$  and  $bi$  lie in the upper half-plane, and none lies on the real axis.

By Theorem 3.3,

$$\int_{-\infty}^{\infty} \frac{1}{(t^2+a^2)(t^2+b^2)} dt = 2\pi i S + \pi i T,$$

where  $T = 0$  and

$$\begin{aligned}
 S &= \text{Res}(p/q, ai) + \text{Res}(p/q, bi) \\
 &= \frac{1}{2ai \cdot (-a^2+b^2)} + \frac{1}{(-b^2+a^2) \cdot 2bi} \\
 &\quad (\text{by the } g/h \text{ Rule}).
 \end{aligned}$$

Thus

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{1}{(t^2+a^2)(t^2+b^2)} dt &= 2\pi i \left( \frac{1}{2ia(b^2-a^2)} + \frac{1}{2ib(a^2-b^2)} \right) \\
 &= \frac{\pi}{ab(a+b)}.
 \end{aligned}$$

**3.11** Let  $p(z) = 1$  and  $q(z) = z$ ; then Conditions 1 and 2 of Theorem 3.4 are satisfied. The only singularity of the function  $f(z) = e^{ikz}/z$  is a simple pole at 0.

By Theorem 3.4,

$$\int_{-\infty}^{\infty} \frac{e^{ikt}}{t} dt = 2\pi i S + \pi i T,$$

where  $S = 0$ , and

$$\begin{aligned}
 T &= \text{Res}(f, 0) \\
 &= \frac{e^0}{1} = 1 \quad (\text{by the } g/h \text{ Rule}).
 \end{aligned}$$

$$\text{Thus } \int_{-\infty}^{\infty} \frac{e^{ikt}}{t} dt = i\pi.$$

Hence, since  $e^{ikt} = \cos kt + i \sin kt$ ,

$$\int_{-\infty}^{\infty} \frac{\cos kt}{t} dt = \text{Re} \left( \int_{-\infty}^{\infty} \frac{e^{ikt}}{t} dt \right) = 0$$

and

$$\int_{-\infty}^{\infty} \frac{\sin kt}{t} dt = \text{Im} \left( \int_{-\infty}^{\infty} \frac{e^{ikt}}{t} dt \right) = \pi.$$



**3.12** Let  $p(z) = 1$  and  $q(z) = z^2 + 9$ ; then Conditions 1 and 2 of Theorem 3.4 are satisfied. The only singularities of the function  $f(z) = e^{2iz}/(z^2 + 9)$  are simple poles at  $3i$  and  $-3i$ .

By Theorem 3.4,

$$\int_{-\infty}^{\infty} \frac{e^{2it}}{t^2 + 9} dt = 2\pi i S + \pi i T,$$

where  $T = 0$ , and

$$\begin{aligned} S &= \text{Res}(f, 3i) \\ &= \frac{e^{-6}}{6i} \quad (\text{by the } g/h \text{ Rule}). \end{aligned}$$

Thus

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{2it}}{t^2 + 9} dt &= 2\pi i \left( \frac{e^{-6}}{6i} \right) \\ &= \frac{\pi}{3e^6}. \end{aligned}$$

Hence, since  $e^{2it} = \cos 2t + i \sin 2t$ ,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos 2t}{t^2 + 9} dt &= \text{Re} \left( \int_{-\infty}^{\infty} \frac{e^{2it}}{t^2 + 9} dt \right) \\ &= \frac{\pi}{3e^6}. \end{aligned}$$

## Section 4

**4.1** The function  $\phi(z) = 1/z^4$  is even and analytic on  $\mathbb{C}$  apart from a pole at 0.

The residue of the function

$$f(z) = (\pi \cot \pi z)/z^4$$

at 0 may be obtained from the Laurent series about 0 for  $f$ :

$$\begin{aligned} \frac{\pi \cot \pi z}{z^4} &= \frac{\pi}{z^4} \left( \frac{1}{\pi z} - \frac{1}{3}(\pi z) - \frac{1}{45}(\pi z)^3 - \dots \right), \\ &\quad \text{using Equation (4.2),} \\ &= \frac{1}{z^5} - \frac{1}{3} \frac{\pi^2}{z^3} - \frac{1}{45} \frac{\pi^4}{z} - \dots; \end{aligned}$$

hence  $\text{Res}(f, 0) = -\pi^4/45$ .

We now check Condition (4.1). If  $z$  lies on  $S_N$ , then  $|z| \geq N + \frac{1}{2}$ , and so, by Lemma 4.1,

$$|f(z)| = \left| \frac{\pi \cot \pi z}{z^4} \right| \leq \frac{2\pi}{(N + \frac{1}{2})^4}, \quad \text{for } z \in S_N.$$

Hence, by the Estimation Theorem,

$$\left| \int_{S_N} f(z) dz \right| \leq \frac{2\pi}{(N + \frac{1}{2})^4} \cdot 4(2N + 1) = \frac{128\pi}{(2N + 1)^3},$$

which tends to 0 as  $N \rightarrow \infty$ . Thus, Condition (4.1) holds.

It follows from Theorem 4.1 that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = -\frac{1}{2} \text{Res}(f, 0) = \frac{\pi^4}{90}.$$

**4.2** The function  $\phi(z) = 1/(4z^2 - 1)$  is even and analytic on  $\mathbb{C}$  apart from simple poles at  $\frac{1}{2}, -\frac{1}{2}$ .

The residues of the function

$$f(z) = (\pi \cot \pi z)/(4z^2 - 1)$$

at  $\frac{1}{2}$  and  $-\frac{1}{2}$  are both 0 (see Problem 1.5(b)).

Since  $\phi$  is analytic at 0,

$$\text{Res}(f, 0) = \phi(0) = -1.$$

We now check Condition (4.1). If  $z$  lies on  $S_N$ , then  $|z| \geq N + \frac{1}{2}$ , and so, by Lemma 4.1 and the Triangle Inequality,

$$|f(z)| = \left| \frac{\pi \cot \pi z}{4z^2 - 1} \right| \leq \frac{2\pi}{4(N + \frac{1}{2})^2 - 1}, \quad \text{for } z \in S_N.$$

Hence, by the Estimation Theorem,

$$\left| \int_{S_N} f(z) dz \right| \leq \frac{2\pi}{4(N + \frac{1}{2})^2 - 1} \cdot 4(2N + 1),$$

which tends to 0 as  $N \rightarrow \infty$ . Thus, Condition (4.1) holds.

It follows from Theorem 4.1 that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} &= -\frac{1}{2} (\text{Res}(f, 0) + \text{Res}(f, \frac{1}{2}) + \text{Res}(f, -\frac{1}{2})) \\ &= \frac{1}{2} + 0 + 0 = \frac{1}{2}. \end{aligned}$$

**4.3** The function  $\phi(z) = 1/z^2$  is even and analytic on  $\mathbb{C}$  apart from a pole at 0.

The residue at 0 of the function

$$f(z) = (\pi \text{cosec } \pi z)/z^2$$

may be obtained from the Laurent series about 0 for  $f$ :

$$\begin{aligned} \frac{(\pi \text{cosec } \pi z)}{z^2} &= \frac{\pi}{z^2} \left( \frac{1}{\pi z} + \frac{\pi z}{6} + \frac{7}{360}(\pi z)^3 + \dots \right), \\ &\quad \text{using Equation (4.7),} \\ &= \frac{1}{z^3} + \frac{\pi^2}{6} \frac{1}{z} + \frac{7\pi^4}{360} z + \dots; \end{aligned}$$

hence  $\text{Res}(f, 0) = \pi^2/6$ .

We now check Condition (4.6). If  $z$  lies on  $S_N$ , then  $|z| \geq N + \frac{1}{2}$ , and so, by Lemma 4.2,

$$|f(z)| = \left| \frac{\pi \text{cosec } \pi z}{z^2} \right| \leq \frac{\pi}{(N + \frac{1}{2})^2}, \quad \text{for } z \in S_N.$$

Hence, by the Estimation Theorem,

$$\left| \int_{S_N} f(z) dz \right| \leq \frac{\pi}{(N + \frac{1}{2})^2} \cdot 4(2N + 1) = \frac{16\pi}{2N + 1},$$

which tends to 0 as  $N \rightarrow \infty$ . Thus, Condition (4.6) holds.

It follows from Theorem 4.2 that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{1}{2} \text{Res}(f, 0) = -\frac{\pi^2}{12}.$$

# SOLUTIONS TO THE EXERCISES

## Section 1

**1.1 (a)** The Laurent series about 0 for the function  $f(z) = e^z/z^7$  is

$$\begin{aligned} \frac{1}{z^7} \left( 1 + z + \frac{z^2}{2!} + \cdots \right) \\ = \cdots + \frac{1}{5!z^2} + \frac{1}{6!z} + \frac{1}{7!} + \cdots, \end{aligned}$$

and so  $\text{Res}(f, 0) = 1/6! = 1/720$ .

Alternatively, applying Theorem 1.2 with  $k = 7$ , we obtain

$$\begin{aligned} \text{Res}(f, 0) &= \frac{1}{6!} \lim_{z \rightarrow 0} \left( \frac{d^6}{dz^6} \left( z^7 \cdot \frac{e^z}{z^7} \right) \right) \\ &= \frac{1}{6!} \lim_{z \rightarrow 0} e^z = \frac{1}{6!}. \end{aligned}$$

**(b)** Substituting  $z - \pi/2 = h$ , so that  $z = \pi/2 + h$ , we obtain, for  $z \neq \pi/2$ ,

$$\begin{aligned} f(z) &= \frac{\cos z}{(z - \pi/2)^2} \\ &= \frac{\cos(\pi/2 + h)}{h^2} \\ &= -\frac{\sin h}{h^2} \\ &= -\frac{1}{h^2} \left( h - \frac{h^3}{3!} + \cdots \right) \\ &= -\frac{1}{h} + \frac{h}{3!} - \cdots. \end{aligned}$$

Hence  $\text{Res}(f, \pi/2) = -1$ .

Alternatively, applying Theorem 1.2 with  $k = 2$ , we obtain

$$\begin{aligned} \text{Res}(f, \pi/2) &= \lim_{z \rightarrow \pi/2} \left( \frac{d}{dz} \left( (z - \pi/2)^2 \frac{\cos z}{(z - \pi/2)^2} \right) \right) \\ &= \lim_{z \rightarrow \pi/2} (-\sin z) \\ &= -1. \end{aligned}$$

**(c)** We use the  $g/h$  Rule with  $g(z) = e^z$ ,  $h(z) = z^4 - 1$  and  $h'(z) = 4z^3$ . (The functions  $g$  and  $h$  are entire,  $h(\alpha) = 0$  and  $h'(\alpha) \neq 0$  for  $\alpha = 1, -1, i, -i$ .)

Thus

$$\begin{aligned} \text{Res}(f, 1) &= \frac{g(1)}{h'(1)} = \frac{e^1}{4 \cdot 1^3} = \frac{1}{4}e; \\ \text{Res}(f, -1) &= \frac{g(-1)}{h'(-1)} = \frac{e^{-1}}{4 \cdot (-1)^3} = -\frac{1}{4}e^{-1}; \\ \text{Res}(f, i) &= \frac{g(i)}{h'(i)} = \frac{e^i}{4i^3} = \frac{1}{4}ie^i; \\ \text{Res}(f, -i) &= \frac{g(-i)}{h'(-i)} = \frac{e^{-i}}{4 \cdot (-i)^3} = -\frac{1}{4}ie^{-i}. \end{aligned}$$

**(d)** Here we use the Cover-up Rule. The function  $f(z) = 1/(z^2 - 4)$  has simple poles at 2 and -2 and

$$f(z) = \frac{1}{(z-2)(z+2)}.$$

Hence we obtain

$$\begin{aligned} \text{Res}(f, 2) &= \frac{1}{2+2} = \frac{1}{4}; \\ \text{Res}(f, -2) &= \frac{1}{-2-2} = -\frac{1}{4}. \end{aligned}$$

**(e)** Since  $f(z)$  can be written as

$$f(z) = \frac{e^z}{z^3(z-3)(z+3)},$$

it is easy to use the Cover-up Rule to find the residues at the simple poles 3 and -3. We obtain

$$\begin{aligned} \text{Res}(f, 3) &= \frac{e^3}{3^3(3+3)} = \frac{e^3}{162}; \\ \text{Res}(f, -3) &= \frac{e^{-3}}{(-3)^3(-3-3)} = \frac{e^{-3}}{162}. \end{aligned}$$

Now  $f$  has a pole of order three at 0; we shall find  $\text{Res}(f, 0)$  from the Laurent series about 0 for  $f$ .

Since

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots, \quad \text{for } z \in \mathbb{C},$$

and

$$\begin{aligned} \frac{1}{z^2 - 9} &= -\frac{1}{9} \cdot \frac{1}{1 - z^2/9} \\ &= -\frac{1}{9} \left( 1 + \frac{z^2}{9} + \left( \frac{z^2}{9} \right)^2 + \cdots \right), \quad \text{for } |z| < 3, \end{aligned}$$

we have

$$f(z) = -\frac{1}{9z^3} \left( 1 + z + \frac{z^2}{2!} + \cdots \right) \left( 1 + \frac{z^2}{9} + \frac{z^4}{81} + \cdots \right),$$

for  $0 < |z| < 3$ .

Since we need only find the coefficient of  $z^{-1}$  in this Laurent series, we determine the coefficient of  $z^2$  in the product of the two brackets. This coefficient is

$$\frac{1}{9} + \frac{1}{2!} = \frac{11}{18},$$

and so

$$\text{Res}(f, 0) = -\frac{1}{9} \cdot \frac{11}{18} = -\frac{11}{162}.$$

(It is also possible to use Theorem 1.2 with  $k = 3$  to find  $\text{Res}(f, 0)$ .)

## Section 2

**2.1 (a)** The function  $f(z) = (\cos z)/(z - \pi/2)^2$  is analytic on  $\mathbb{C}$  apart from a singularity at  $\pi/2$ , which lies inside  $C = \{z : |z| = 2\}$ . Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$  and Exercise 1.1(b),

$$\int_C \frac{\cos z}{(z - \pi/2)^2} dz = 2\pi i(-1) = -2\pi i.$$

**(b)** The function  $f(z) = e^z/(z^4 - 1)$  is analytic on  $\mathbb{C}$  apart from singularities at 1, -1,  $i$  and  $-i$ , all of which lie inside  $C = \{z : |z| = 2\}$ . Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$  and Exercise 1.1(c),

$$\begin{aligned} \int_C \frac{e^z}{z^4 - 1} dz &= 2\pi i \left( \frac{1}{4}e - \frac{1}{4}e^{-1} + \frac{1}{4}ie^i - \frac{1}{4}ie^{-i} \right) \\ &= \pi i(\sinh 1 - \sin 1). \end{aligned}$$

**(c)** The function  $f(z) = e^z/(z^3(z^2 - 9))$  is analytic on  $\mathbb{C}$  apart from singularities at 0, 3, -3. Of these, only 0 lies inside  $C = \{z : |z| = 2\}$ ; neither of the other two lies on  $C$ . Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$  and Exercise 1.1(e),

$$\int_C \frac{e^z}{z^3(z^2 - 9)} dz = 2\pi i \left( -\frac{11}{162} \right) = -\frac{11\pi i}{81}.$$



**2.2** The function  $f(z) = z/(e^z - 1)$  is analytic on  $\mathbb{C}$  apart from singularities at the points  $2\pi ki$ , for  $k \in \mathbb{Z}$  (the zeros of  $z \mapsto e^z - 1$ ).

Using the  $g/h$  Rule, with  $g(z) = z$ ,  $h(z) = e^z - 1$  and  $h'(z) = e^z$ , we obtain

$$\operatorname{Res}(f, 2k\pi i) = \frac{2k\pi i}{e^{2k\pi i}} = 2k\pi i, \quad \text{for } k \in \mathbb{Z}.$$

(a) The only singularity of  $f$  inside  $C = \{z : |z| = 1\}$  is at 0; none of the other singularities lies on  $C$ . Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ ,

$$\int_C \frac{z}{e^z - 1} dz = 2\pi i \operatorname{Res}(f, 0) = 0.$$

(This result is as expected since  $f$  has a removable singularity at 0 which, when removed, makes  $f$  analytic on  $\{z : |z| < 2\pi\}$  and so Cauchy's Theorem may be applied.)

(b) The singularities of  $f$  inside  $C = \{z : |z - 3i| = 4\}$  are those at 0 and  $2\pi i$ ; none of the other singularities lies on  $C$ . Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ ,

$$\begin{aligned} \int_C \frac{z}{e^z - 1} dz &= 2\pi i (\operatorname{Res}(f, 0) + \operatorname{Res}(f, 2\pi i)) \\ &= 2\pi i (0 + 2\pi i) = -4\pi^2. \end{aligned}$$

**2.3** Following the strategy, we obtain

$$\begin{aligned} \int_0^{2\pi} \frac{1}{4 \cos^2 t + \sin^2 t} dt \\ &= \int_C \frac{1}{4((z + z^{-1})/2)^2 + ((z - z^{-1})/2i)^2} \cdot \frac{1}{iz} dz \\ &= \frac{4}{i} \int_C \frac{z}{3z^4 + 10z^2 + 3} dz, \end{aligned}$$

where  $C = \{z : |z| = 1\}$ .

Since  $3z^4 + 10z^2 + 3 = (z^2 + 3)(3z^2 + 1)$ , the singularities of the function

$$f(z) = \frac{4}{i} \cdot \frac{z}{3z^4 + 10z^2 + 3}$$

are simple poles at  $i\sqrt{3}$ ,  $-i\sqrt{3}$ ,  $i/\sqrt{3}$  and  $-i/\sqrt{3}$ . Of these, only  $i/\sqrt{3}$  and  $-i/\sqrt{3}$  lie inside  $C$ , and neither of the other two lies on  $C$ .

Using the  $g/h$  Rule with  $g(z) = 4z/i$ ,  $h(z) = 3z^4 + 10z^2 + 3$  and  $h'(z) = 12z^3 + 20z$ , we obtain

$$\operatorname{Res}(f, i/\sqrt{3}) = \frac{4(i/\sqrt{3})/i}{12(i/\sqrt{3})^3 + 20(i/\sqrt{3})} = -\frac{1}{4}i;$$

$$\operatorname{Res}(f, -i/\sqrt{3}) = \frac{4(-i/\sqrt{3})/i}{12(-i/\sqrt{3})^3 + 20(-i/\sqrt{3})} = -\frac{1}{4}i.$$

Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ ,

$$\frac{4}{i} \int_C \frac{z}{3z^4 + 10z^2 + 3} dz = 2\pi i \left(-\frac{1}{4}i - \frac{1}{4}i\right) = \pi,$$

and so

$$\int_0^{2\pi} \frac{1}{4 \cos^2 t + \sin^2 t} dt = \pi.$$

**2.4** If  $a > 1$ , then the singularities of the function

$$f(z) = \frac{1}{z^2 - 2az + 1}$$

are at  $z = a \pm \sqrt{a^2 - 1}$ . Clearly  $a + \sqrt{a^2 - 1} > 1$ , so that  $a + \sqrt{a^2 - 1}$  lies outside  $C = \{z : |z| = 1\}$ .

On the other hand, since

$$(a - \sqrt{a^2 - 1})(a + \sqrt{a^2 - 1}) = a^2 - (a^2 - 1) = 1,$$

we have

$$0 < a - \sqrt{a^2 - 1} < 1$$

and so  $a - \sqrt{a^2 - 1}$  lies inside  $C$ .

Using the  $g/h$  Rule, with  $g(z) = 1$ ,  $h(z) = z^2 - 2az + 1$  and  $h'(z) = 2z - 2a$  (or the Cover-up Rule), we obtain

$$\begin{aligned} \operatorname{Res}\left(f, a - \sqrt{a^2 - 1}\right) &= \frac{1}{2(a - \sqrt{a^2 - 1}) - 2a} \\ &= -\frac{1}{2\sqrt{a^2 - 1}}. \end{aligned}$$

Hence, by the Residue Theorem with  $\mathcal{R} = \mathbb{C}$ ,

$$\begin{aligned} \int_C \frac{1}{z^2 - 2az + 1} dz &= 2\pi i \left(-\frac{1}{2\sqrt{a^2 - 1}}\right) \\ &= \frac{-\pi i}{\sqrt{a^2 - 1}}, \quad \text{as required.} \end{aligned}$$

Applying the strategy, we obtain

$$\begin{aligned} \int_0^{2\pi} \frac{1}{a - \cos t} dt &= \int_C \frac{1}{a - (z + z^{-1})/2} \cdot \frac{1}{iz} dz \\ &= \int_C \frac{2i}{z^2 - 2az + 1} dz \\ &= 2i \left(-\frac{\pi i}{\sqrt{a^2 - 1}}\right) = \frac{2\pi}{\sqrt{a^2 - 1}}. \end{aligned}$$

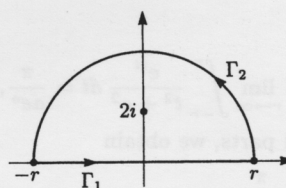
## Section 3

**3.1** In this answer we have not labelled the steps of the strategy.

(a) Consider the contour integral

$$I = \int_{\Gamma} \frac{z^2}{(z^2 + 4)^2} dz,$$

where  $\Gamma = \Gamma_1 + \Gamma_2$  is the contour shown in the figure.



The function  $f(z) = z^2/(z^2 + 4)^2$  is analytic on  $\mathbb{C}$  apart from poles of order 2 at  $2i$ ,  $-2i$ . Of these  $2i$  lies inside  $\Gamma$ , for  $r > 2$ , and  $-2i$  lies outside  $\Gamma$ . Now,

$$\begin{aligned} \operatorname{Res}(f, 2i) &= \lim_{z \rightarrow 2i} \left( \frac{d}{dz} \left( \frac{(z - 2i)^2 z^2}{(z^2 + 4)^2} \right) \right) \\ &= \lim_{z \rightarrow 2i} \left( \frac{d}{dz} \left( \frac{z^2}{(z + 2i)^2} \right) \right) \\ &= \lim_{z \rightarrow 2i} \left( \frac{(z + 2i)^2 \cdot 2z - z^2 \cdot 2(z + 2i)}{(z + 2i)^4} \right) \\ &= -\frac{1}{8}i. \end{aligned}$$

Hence, by the Residue Theorem,

$$I = 2\pi i \left(-\frac{1}{8}i\right) = \pi/4,$$

and so

$$I = \int_{-r}^r \frac{t^2}{(t^2 + 4)^2} dt + \int_{\Gamma_2} \frac{z^2}{(z^2 + 4)^2} dz = \frac{\pi}{4}.$$

By the Estimation Theorem and the Triangle Inequality,

$$\left| \int_{\Gamma_2} \frac{z^2}{(z^2 + 4)^2} dz \right| \leq \frac{r^2}{(r^2 - 4)^2} \times \pi r = \frac{\pi r^3}{(r^2 - 4)^2}, \quad \text{for } r > 2,$$

and so

$$\lim_{r \rightarrow \infty} \int_{\Gamma_2} \frac{z^2}{(z^2 + 4)^2} dz = 0.$$

Hence

$$\int_{-\infty}^{\infty} \frac{t^2}{(t^2 + 4)^2} dt = \lim_{r \rightarrow \infty} \int_{-r}^r \frac{t^2}{(t^2 + 4)^2} dt = \frac{\pi}{4}.$$

(b) Consider the contour integral

$$I = \int_{\Gamma} \frac{e^{iz}}{z^2 + a^2} dz,$$

where  $\Gamma = \Gamma_1 + \Gamma_2$  is the contour shown in part (a).

The function  $f(z) = e^{iz}/(z^2 + a^2)$  is analytic on  $\mathbb{C}$  apart from simple poles at  $ai, -ai$ . Of these,  $ai$  lies inside  $\Gamma$ , for  $r > a$ , and  $-ai$  lies outside  $\Gamma$ . Now

$$\text{Res}(f, ai) = \frac{e^{i(ai)}}{2ai} = \frac{e^{-a}}{2ai},$$

by, for example, the  $g/h$  Rule.

Hence, by the Residue Theorem,

$$I = 2\pi i \left( \frac{e^{-a}}{2ai} \right) = \frac{\pi}{ae^a},$$

and so

$$I = \int_{-r}^r \frac{e^{it}}{t^2 + a^2} dt + \int_{\Gamma_2} \frac{e^{iz}}{z^2 + a^2} dz = \frac{\pi}{ae^a}.$$

Now  $|e^{iz}| = e^{-y} \leq 1$ , for  $z = x + iy$  on  $\Gamma_2$ , and so, by the Estimation Theorem and the Triangle Inequality,

$$\left| \int_{\Gamma_2} \frac{e^{iz}}{z^2 + a^2} dz \right| \leq \frac{1}{r^2 - a^2} \times \pi r = \frac{\pi r}{r^2 - a^2}, \quad \text{for } r > a.$$

Thus

$$\lim_{r \rightarrow \infty} \int_{\Gamma_2} \frac{e^{iz}}{z^2 + a^2} dz = 0.$$

Hence

$$\int_{-\infty}^{\infty} \frac{e^{it}}{t^2 + a^2} dt = \lim_{r \rightarrow \infty} \int_{-r}^r \frac{e^{it}}{t^2 + a^2} dt = \frac{\pi}{ae^a},$$

and so, equating real parts, we obtain

$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2 + a^2} dt = \frac{\pi}{ae^a}.$$

**3.2 (a)** By the (real) Fundamental Theorem of Calculus,

$$\int_0^r \frac{1}{(t+1)^2} dt = \left[ -\frac{1}{(t+1)} \right]_0^r = 1 - \frac{1}{r+1}.$$

Since  $1/(r+1) \rightarrow 0$  as  $r \rightarrow \infty$ , we deduce that

$$\int_0^{\infty} \frac{1}{(t+1)^2} dt = \lim_{r \rightarrow \infty} \int_0^r \frac{1}{(t+1)^2} dt = 1.$$

(b) Since the integrand is an odd function, we deduce from Theorem 3.2(a) that

$$\int_{-\infty}^{\infty} \frac{t}{t^4 + 1} dt = 0.$$

(c) From Exercise 3.1(a), we know that

$$\int_{-\infty}^{\infty} \frac{t^2}{(t^2 + 4)^2} dt = \frac{\pi}{4}.$$

Since the integrand is an even function, we deduce from Theorem 3.2(b) that

$$\int_0^{\infty} \frac{t^2}{(t^2 + 4)^2} dt = \frac{1}{2} \int_{-\infty}^{\infty} \frac{t^2}{(t^2 + 4)^2} dt = \frac{\pi}{8}.$$

(d) We apply Theorem 3.3 with  $p(z) = z$  and  $q(z) = z^3 + 1$ . This is permissible since the degree of  $q$  exceeds that of  $p$  by 2 and  $q$  has one zero on the real axis, at  $-1$ , which is simple.

The singularities of the function  $p/q$  are simple poles at

$$\frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i \text{ and } -1,$$

of which that at  $\frac{1}{2} + \frac{\sqrt{3}}{2}i$  lies in the upper half-plane and that at  $-1$  lies on the real axis. Now, using the  $g/h$  Rule with  $g(z) = p(z) = z$ ,  $h(z) = q(z) = z^3 + 1$  and  $h'(z) = 3z^2$ , we obtain

$$\text{Res}(f, -1) = \frac{-1}{3(-1)^2} = -\frac{1}{3};$$

$$\begin{aligned} \text{Res}\left(f, \frac{1}{2} + \frac{\sqrt{3}}{2}i\right) &= \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}i}{3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2} \\ &= \frac{1}{3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)} = \frac{1}{3}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right). \end{aligned}$$

Hence, by Theorem 3.3,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{t}{t^3 + 1} dt &= 2\pi i S + \pi i T \\ &= 2\pi i \left( \frac{1}{3} \left( \frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \right) + \pi i \left( -\frac{1}{3} \right) \\ &= 2\pi i \left( -\frac{\sqrt{3}}{6}i \right) = \frac{\pi}{\sqrt{3}}. \end{aligned}$$

(e) We apply Theorem 3.4 with  $p(z) = z$ ,  $q(z) = 1 - z^2$  and  $k = \pi$ . This is permissible since the degree of  $q$  exceeds that of  $p$  by 1, and  $q$  has two zeros on the real axis, at 1 and  $-1$ , which are simple.

The singularities of the function

$$f(z) = \frac{p(z)e^{i\pi z}}{q(z)} = \frac{ze^{i\pi z}}{1 - z^2}$$

are simple poles at 1 and  $-1$ , which lie on the real axis.

Now, using the  $g/h$  Rule with  $g(z) = ze^{i\pi z}$ ,  $h(z) = q(z) = 1 - z^2$  and  $h'(z) = -2z$ , we obtain

$$\text{Res}(f, 1) = \frac{e^{i\pi}}{-2} = \frac{1}{2};$$

$$\text{Res}(f, -1) = \frac{-e^{-i\pi}}{2} = \frac{1}{2}.$$

Hence, by Theorem 3.4,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{te^{i\pi t}}{1 - t^2} dt &= 2\pi i S + \pi i T \\ &= 2\pi i \cdot 0 + \pi i \left( \frac{1}{2} + \frac{1}{2} \right) \\ &= \pi i. \end{aligned}$$

Thus, equating imaginary parts, we obtain

$$\int_{-\infty}^{\infty} \frac{t \sin \pi t}{1 - t^2} dt = \pi.$$



## Section 4

**4.1 (a)** The function  $\phi(z) = 1/(4z^2 + 1)$  is even and analytic on  $\mathbb{C}$  apart from simple poles at  $\frac{1}{2}i, -\frac{1}{2}i$ .

The residues of the function

$$f(z) = (\pi \cot \pi z)/(4z^2 + 1)$$

at  $\frac{1}{2}i$  and  $-\frac{1}{2}i$  were found in Problem 1.5(d):

$$\operatorname{Res}(f, \frac{1}{2}i) = \operatorname{Res}(f, -\frac{1}{2}i) = -\frac{\pi \cosh \frac{1}{2}\pi}{4 \sinh \frac{1}{2}\pi}.$$

Since  $\phi$  is analytic at 0,

$$\operatorname{Res}(f, 0) = \phi(0) = 1.$$

We now check Condition (4.1). If  $z$  lies on  $S_N$ , then  $|z| \geq N + \frac{1}{2}$ , and so, by Lemma 4.1 and the Triangle Inequality,

$$|f(z)| = \left| \frac{\pi \cot \pi z}{4z^2 + 1} \right| \leq \frac{2\pi}{4(N + \frac{1}{2})^2 - 1}, \quad \text{for } z \in S_N.$$

Hence, by the Estimation Theorem,

$$\left| \int_{S_N} f(z) dz \right| \leq \frac{2\pi}{4(N + \frac{1}{2})^2 - 1} \cdot 4(2N + 1),$$

which tends to 0 as  $N \rightarrow \infty$ . Thus, Condition (4.1) holds.

It follows from Theorem 4.1 that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{4n^2 + 1} &= -\frac{1}{2} (\operatorname{Res}(f, 0) + \operatorname{Res}(f, \frac{1}{2}i) + \operatorname{Res}(f, -\frac{1}{2}i)) \\ &= -\frac{1}{2} \left( 1 - \frac{\pi \cosh \frac{1}{2}\pi}{4 \sinh \frac{1}{2}\pi} - \frac{\pi \cosh \frac{1}{2}\pi}{4 \sinh \frac{1}{2}\pi} \right) \\ &= \frac{\pi \cosh \frac{1}{2}\pi}{4 \sinh \frac{1}{2}\pi} - \frac{1}{2}. \end{aligned}$$

**(b)** The function  $\phi(z) = 1/(4z^2 + 1)$  is even and analytic on  $\mathbb{C}$  apart from simple poles at  $\frac{1}{2}i, -\frac{1}{2}i$ .

The residues of the function

$$f(z) = (\pi \operatorname{cosec} \pi z)/(4z^2 + 1)$$

at  $\frac{1}{2}i$  and  $-\frac{1}{2}i$  were found in Problem 1.5(c):

$$\operatorname{Res}(f, \frac{1}{2}i) = \operatorname{Res}(f, -\frac{1}{2}i) = -\frac{\pi}{4 \sinh \frac{1}{2}\pi}.$$

Since  $\phi$  is analytic at 0,

$$\operatorname{Res}(f, 0) = \phi(0) = 1.$$

We now check Condition (4.6). If  $z$  lies on  $S_N$ , then  $|z| \geq N + \frac{1}{2}$ , and so, by Lemma 4.2 and the Triangle Inequality,

$$|f(z)| = \left| \frac{\pi \operatorname{cosec} \pi z}{4z^2 + 1} \right| \leq \frac{\pi}{4(N + \frac{1}{2})^2 - 1}, \quad \text{for } z \in S_N.$$

Hence, by the Estimation Theorem,

$$\left| \int_{S_N} f(z) dz \right| \leq \frac{\pi}{4(N + \frac{1}{2})^2 - 1} \cdot 4(2N + 1),$$

which tends to 0 as  $N \rightarrow \infty$ . Thus Condition (4.6) holds.

It follows from Theorem 4.2 that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 + 1} &= -\frac{1}{2} (\operatorname{Res}(f, 0) + \operatorname{Res}(f, \frac{1}{2}i) + \operatorname{Res}(f, -\frac{1}{2}i)) \\ &= -\frac{1}{2} \left( 1 - \frac{\pi}{4 \sinh \frac{1}{2}\pi} - \frac{\pi}{4 \sinh \frac{1}{2}\pi} \right) \\ &= \frac{\pi}{4 \sinh \frac{1}{2}\pi} - \frac{1}{2}. \end{aligned}$$

**4.2** Let  $\alpha \in \mathbb{C} - \mathbb{Z}$ . Then the function

$\phi(z) = 1/(z^2 - \alpha^2)$  is even and analytic on  $\mathbb{C}$  apart from simple poles at  $\alpha$  and  $-\alpha$ , which are not integers.

We now calculate the residues of the function

$$f(z) = (\pi \cot \pi z)/(z^2 - \alpha^2)$$

at 0,  $\alpha$  and  $-\alpha$ .

Since  $\phi$  is analytic at 0,

$$\operatorname{Res}(f, 0) = \phi(0) = -1/\alpha^2.$$

Also, using the  $g/h$  Rule with  $g(z) = \pi \cot \pi z$ ,  $h(z) = z^2 - \alpha^2$  and  $h'(z) = 2z$ , we obtain

$$\operatorname{Res}(f, \alpha) = \frac{\pi \cot \pi \alpha}{2\alpha};$$

$$\operatorname{Res}(f, -\alpha) = \frac{\pi \cot(-\pi \alpha)}{-2\alpha} = \frac{\pi \cot \pi \alpha}{2\alpha}.$$

We now check Condition (4.1). If  $z$  lies on  $S_N$ , then  $|z| \geq N + \frac{1}{2}$ , and so, by Lemma 4.1 and the Triangle Inequality,

$$\begin{aligned} |f(z)| &= \left| \frac{\pi \cot \pi z}{z^2 - \alpha^2} \right| \\ &\leq \frac{2\pi}{|z|^2 - |\alpha|^2}, \quad \text{for } |z| > |\alpha| \\ &\leq \frac{2\pi}{(N + \frac{1}{2})^2 - |\alpha|^2}, \quad \text{for } z \in S_N, N + \frac{1}{2} > |\alpha|. \end{aligned}$$

Hence, by the Estimation Theorem,

$$\left| \int_{S_N} f(z) dz \right| \leq \frac{2\pi \cdot 4(2N + 1)}{(N + \frac{1}{2})^2 - |\alpha|^2}, \quad \text{for } N + \frac{1}{2} > |\alpha|,$$

which tends to 0 as  $N \rightarrow \infty$ . Thus Condition (4.1) holds.

It follows from Theorem 4.1 that

$$\begin{aligned} \sum_{n=1}^{\infty} \phi(n) &= \sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} \\ &= -\frac{1}{2} (\operatorname{Res}(f, 0) + \operatorname{Res}(f, \alpha) + \operatorname{Res}(f, -\alpha)) \\ &= -\frac{1}{2} \left( -\frac{1}{\alpha^2} + \frac{\pi \cot \pi \alpha}{2\alpha} + \frac{\pi \cot \pi \alpha}{2\alpha} \right) \\ &= \frac{1}{2\alpha^2} - \frac{\pi \cot \pi \alpha}{2\alpha}. \end{aligned}$$

Rearranging this, we obtain

$$\begin{aligned} \pi \cot \pi \alpha &= \frac{1}{\alpha} - 2\alpha \sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} \\ &= \frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2 - n^2}, \end{aligned} \quad (1)$$

as required.

*Remark* Notice that Equation (1) can be written in the form

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}, \quad \text{for } z \in \mathbb{C} - \mathbb{Z},$$

which gives a representation of the function  $z \mapsto \pi \cot \pi z$  as the sum of an infinite series of rational functions, valid for all points in the domain of this function.